The invention of random variables
Concept and name

Workshop on History of Statistics
Department of Statistics
Ludwig-Maximilians-Universität München
March 22, 2016

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By tracing the history of the names *variabile casuale*, *zufällige Variable*, *variable aléatoire*, and *random variable*, we learn something about the concept as well.

The story features Galton as well as Laplace, and Chuprov, Fréchet and Darmois as well as Markov, Cantelli, Neyman, Wald, and Doob.

Related working paper:

When to call a variable random, 2015

Wikipedia, 18 March 2016

1. In der Stochastik ist eine **Zufallsvariable** oder **Zufallsgröße** (auch **zufällige Größe**, **Zufallsveränderliche**, selten **stochastische Variable** oder **stochastische Größe**) eine Größe, deren Wert vom Zufall abhängig ist.
2. En théorie des probabilités, une **variable aléatoire** est une application définie sur l'ensemble des éventualités, c'est-à-dire l'ensemble des résultats possibles d'une expérience aléatoire.
3. In probability and statistics, a **random variable**, **random quantity**, **aleatory variable** or **stochastic variable** is a variable whose value is subject to variations due to chance (i.e. randomness, in a mathematical sense).
4. In matematica, e in particolare nella teoria della probabilità, una **variabile casuale** (detta anche **variabile aleatoria** o **variabile stocastica**) è una variabile che può assumere valori diversi in dipendenza da qualche fenomeno aleatorio.
5. **Случайная величина** является одним из основных понятий теории вероятностей.
6. En **stokastisk variabel** (eller **slumpvariabel**) är ett matematiskt objekt som är avsett att beskriva något som påverkas av slumpen.
7. In de kansrekening is een **stochastische variabele** een grootheid waarvan de waarde een reëel getal is dat afhangt van de toevallige uitkomst in een kansexperiment.
8. **Zmienna losowa** – funkcja przypisująca zdarzeniom elementarnym liczby.

Wikipedia, 18 March 2016

1. Der **Erwartungswert** (selten und doppeldeutig *Mittelwert*) ist ein Grundbegriff der Stochastik. Der Erwartungswert einer Zufallsvariablen beschreibt die Zahl, die die Zufalls-variable im Mittel annimmt.
2. En théorie des probabilités, l'**espérance mathématique** d'une variable aléatoire réelle est, intuitivement, la valeur que l'on s'attend à trouver, en moyenne, si l'on répète un grand nombre de fois la même expérience aléatoire.
3. In probability theory, the **expected value** of a random variable is intuitively the long-run average value of repetitions of the experiment it represents. The expected value is also known as the **expectation**, **mathematical expectation**, **EV**, **average**, **mean value**, **mean**, or **first moment**.
4. In teoria della probabilità il **valore atteso** (chiamato anche **media**, **speranza** o **speranza matematica**) di una variabile casuale...
5. **Математи́ческое ожида́ние** — понятие среднего значения вероятностей.
6. **Väntevärde** är inom matematisk statistik en egenskap hos en stokastisk variabel X och dess sannolikhetsfördelning.
7. In de kansrekening is de **verwachting** (of **verwachtingswaarde**) van een stochastische variabele de waarde die deze stochastische variabele 'gemiddeld genomen' zal aannemen.
8. **Wartość oczekiwana** (**wartość średnia, przeciętna,** dawniej **nadzieja matematyczna**) – wartość określająca spodziewany wynik doświadczenia losowego.

Espérance mathématique & valeur moyenne

For Louis Bachelier in 1914, mathematical expectation and mean value were not quite the same thing.

Louis Bachelier
1623-1662

See *Louis Bachelier's Theory of Speculation: The Origins of Modern Finance.* Translated and with an Introduction by Mark Davis & Alison Etheridge. Princeton, 2006.

Les notions de valeur moyenne et d'espérance mathématique sont analogues et même identiques. ..

Les espérances mathématiques qui sont des sommes d'argent fictives s'ajoutent comme des sommes d'argent ordinaires.

Les valeurs moyennes s'ajoutent également, mais leur propriété d'addition parait moins évidente.

Louis Bachelier, *Le jeu, la chance, et le hasard*. Flammarion, Paris, 1914, p. 58.

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The concepts of mean value and mathematical expectation are analogous, even identical. ...

Mathematical expectations, which are fictional sums of money, add just like ordinary sums of money do.

Mean values also add, but their additivity seems less obvious.

Die Konzepte der Mittelwert und der mathematische Erwartung sind ähnlich und sogar identisch. ...

Mathematische Erwartungen sind fiktive Geldsummen und summieren sich wie gewöhnliche Geldsummen.

Die Mittelwerte werden auch hinzugefügt, aber der Eigenschaft scheint weniger offensichtlich.

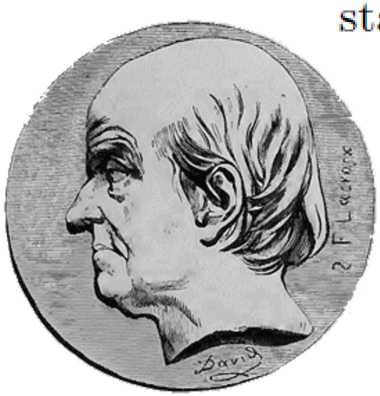
To understand Bachelier 1914, go back a century.

Lacroix's textbook,

- published in 1816
- translated by Unger 1818

La considération des sommes éventuelles, c'est-a-dire dépendantes du hasard, a introduit dans le calcul des probabilités une expression qui mérite un examen particulier, celle de d'*espérance mathématique*, par laquelle on désigne le produit d'une somme éventuelle par la probabilité de l'obtenir.

Die Untersuchung der ungewissen Summen, d. h. solcher, die vom Zufalle abhängen, hat in der Wahrscheinlichkeitsrechnung einen Ausdruck eingeführt, der einer bei sondern Prüfung verdient, nämlich *mathematische Hoffnung* worunter *das Product einer ungewissen Summe mit der Wahrscheinlichkeit sie zu erhalten multiplicirt*, verstanden wird.



Sylvestre-François Lacroix
1765-1843

The consideration of gains that are contingent, i.e., depend on chance, has introduced into the probability calculus an expression that merits examination, that of *mathematical expectation*, by which we mean the *product of a contingent gain by the probability of obtaining it*.

Lacroix: The consideration of gains that are contingent, i.e., depend on chance, has introduced an expression that merits examination into the probability calculus, that of *mathematical expectation*, by which we mean the *product of a contingent gain by the probability of obtaining it*.

Example: Suppose Lacroix will receive $\$X$, where the random variable X is equal to 6, 15, or 30, with probabilities 1/3 each.

Lacroix has three contingent gains (sommes éventuelles), with mathematical expectations 2, 5, and 10, respectively.

Lacroix's total mathematical expectation is 17.

Has Lacroix added the expected values of three dependent random variables?

No. He has merely added three dollar amounts.

QUESTIONS

Between Lacroix in 1816 and Bachelier in 1914, the concepts *random variable* and *mean of a random variable* were invented. Where, when, how?

1. How did a quantity for which we have probabilities get to be a *variable*?
2. How did the meaning of *mathematical expectation* change?
3. How did we get *random variable*, *variable aléatoire*, *Zufallsvariable*, случайная величина, etc.?
4. Why do I care?

Question 1. How did a quantity for which we have probabilities get to be a *variable*?

The concept of a probability distribution came first:

Lagrange 1776: *loi de facilité*

Laplace 1781: *loi de possibilité*

Laplace 1812: *loi de probabilité*

In a 1781 memoir (called to my attention by Hans Fischer), Laplace used *variable*.

As usual (e.g., Descartes, Newton), *quantité variable* is shortened to *variable*.

Soient n quantités variables et positives $t, t_1, t_2, \ldots, t_{n-1}$, dont la somme soit s et dont la loi de possibilité soit connue; on propose de trouver la somme des produits de chaque valeur que peut recevoir une fonction donnée $\psi(t, t_1, t_2, \ldots, t_{n-1})$ de ces variables, multipliée par la probabilité correspondante à cette valeur.[10]

It looks like he is calculating an expected value, but actually he is doing a Bayesian calculation.

Question 1. How did a quantity for which we have probabilities get to be a *variable*?

Laplace's use of *variable* isolated & not followed by others. We do not find it in Gauss.

Some authors contend that Poisson (1837) invented the concepts random variable and expected value.

He did not name them.

…si A est une chose quelconque, qui soit susceptible de différents valeurs à chaque épreuve, la somme de ses valeurs que l'on observera dans une longue série d'épreuves, sera à très peu près et très probablement, proportionnelle à leur nombre. Le rapport de cette somme à ce nombre, pour une chose déterminée A, convergera indéfiniment vers une valeur spéciale qu'il atteindrait si ce nombre pouvait devenir infini, et qui dépend de la loi de probabilité des diverse valeurs possibles de A.[16]

Question 1. How did a quantity for which we have probabilities get to be a *variable*?

Cournot (1843) used *variable* intermittently, along with other names.

He also talked about the *valeur moyenne* and its linearity properties.

He never applied an adjective such as *aléatoire* or *fortuit* to *variable*.

The systematic use of *variable* for an object that takes different values with different probabilities begins with the British statisticians:

- Galton (1887),
- Yule (1895),
- Yule's textbook (1911).

Question 2. How did the meaning of *mathematical expectation* change?

Chebyshev, in his 1867 proof of Poisson's law of large numbers, published in French and Russian, was the first to use **математи́ческое ожида́ние** and **espérance mathématique** for the mean of a random quantity (which he called a **величина** or **grandeur** or **quantité**).

Previously, **математи́ческое ожида́ние** and **espérance mathématique** (and **mathematical expectation** and **mathematische Hoffnung** or – less often – **mathematische Erwartung**) were used only in the way Bachelier was still using **espérance mathématique** in 1914.

Pafnutii Chebyshev
1821-1894

Andrei Markov
1856-1922

Chebyshev's use of **математи́ческое ожида́ние** promoted by

- Bortkiewicz (**mathematische Erwartung)**,
- Chuprov (also **mathematical expectation**),
- Markov (**mathematische Hoffnung** in the 1912 translation of his textbook).

Question 3. How did we get *random variable*, *variable aléatoire*, *Zufallsvariable*, случайная величина, etc.?

In 1898, the Moscow mathematician Pavel Nekrasov became the first to add *random* to *variable*. He used **случайная переменная** (random variable) as well as **случайная величина** (random quantity). Disreputable politically and intellectually, he has been credited with little. But he probably deserves full credit for this bad idea.

Ladislaus Bortkiewicz
1868-1931

Aleksandr Chuprov
1874-1926

Markov was outspoken about not using **случайная**. Markov also followed Chebyshev in always using **величина** (quantity), not **переменная** (variable).

Bortkiewicz and Chuprov did use and promote **случайная**.

- Bortkiewicz 1917: **zufällige Größe**
- Chuprov 1918: **zufällige Variable**
- Chuprov 1925: chance variable

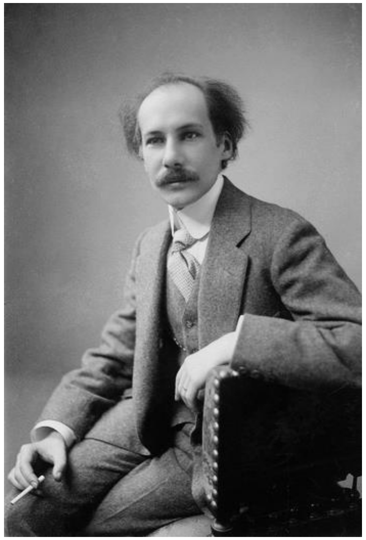
Unlike Bortkiewicz, Chuprov aimed to marry Russian probability with British statistics.

Leaders of the Moscow Mathematical Society

Nikolaus Braschmann
1796-1866

Nikolai Bugaev
1837-1903

Pavel Nekrasov
1853-1924

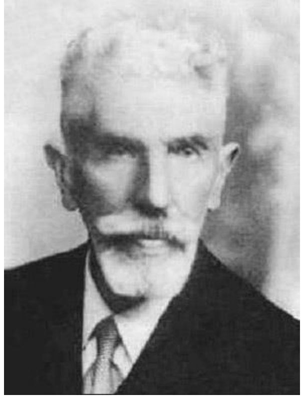


Andrei Bely
1880-1934
Bugaev's son
Symbolist
Author of *Petersburg*

Question 3. How did we get *random variable*, *variable aléatoire*, *Zufallsvariable*, случайная величина, etc.?

Like Chuprov, Cantelli sought to combine Russian advances in probability theory and British advances in statistics.

Francesco Paolo Cantelli
1875-1966



Guido Castelnuovo
1865-1952)

In 1913, Cantelli coined the term **variabile casuale**, which can be translated as **chance variable** or **Zufallsvariable**.

These terms are more logical than Nekrasov's **случайная переменная** (which Cantelli probably never saw) or Chuprov's **случайная переменная** and **zufällige Variable**, which came after 1913.

In 1919, Cantelli's innovations were brought to the attention of European mathematics by a very influential textbook published by his colleague Castelnuovo.

Question 3. How did we get *random variable*, *variable aléatoire*, *Zufallsvariable*, случайная величина, etc.?

In the period 1900-1930, we can find the term random variable from time to time.

But its general use was blocked by two facts:

1. It is illogical: the variable is not random; its values are random.
2. In English: random was understood to mean uniformly distributed.

Russians (Chuprov, Slutskii, Anderson, and Khinchin) writing in German were responsible for the illogical **zufällige Variable**.

Foreigners writing in English (Cramer, Neyman, and Feller) were responsible for the illogical **random variable**.

1912 German translation of Markov's 1900 Russian textbook very influential

One of Markov's innovations: $X,Y,\ldots$ for random variables and $x,y,\ldots$ for their possible values.

Let's look at later textbooks that adopted this innovation.

In terminology Markov followed Chebyshev:

- **математи́ческое ожида́ние**
- **веичина**

The German translations:

- **mathematische Hoffnung**
- **Größe**

Priorities for the notation:

- Markov used **м.о.** X for **математи́ческое ожида́ние**.
- Earlier, Crofton in Britain had used M(X) for **mean**.
- Bortkiewicz, after Markov, had used $E(X)$ for **mathematische Erwartung**.

Cournot never called a variable random. The value is random, not the variable.

Table 1: Nouns that Cournot used most often with *fortuit* in his 1843 *Exposition* [78], with approximate counts of the number of sections where the terms appear, translations used by Schnuse in his 1849 German translation [79], and translations that might be used in teaching probability in English today. (The book had 240 sections altogether.) Schnuse did not translate the entire book, and the omitted passages included those where Cournot used *manière fortuite*.

	Cournot 1843	*Schnuse 1849*	*Possible English*
18	cause fortuite	zufällige Ursache	random cause
10	écart fortuit	zufällige Abweichung	random deviation
8	anomalie fortuite	zufällige Anomalie	random anomaly
8	combinaison fortuite	zufällige Verbindung	random combination
4	rencontre fortuite	zufällig Zusammentreffen	random encounter
4	tirage fortuit	zufällig Zug	random draw
3	épreuve fortuite	zufällig Versuch	trial
3	erreur fortuite	zufällig Fehler	random error
3	événement fortuit	zufällig Ereignis	random event
3	manière fortuite	—	random manner
3	valeur fortuite	zufällig Wert	random value

Until the 20th century, *aléatoire* was used only for financial contingencies (gambling, contracts, inheritance).

The *événement aléatoire* was the event that produced the payoff.

Table 2: Nouns that Cournot used with *aléatoire* in his 1843 *Exposition* [78], with approximate counts of the number of sections where the terms appear, translations used by Schnuse in his 1849 German translation [79], and translations that might be used in teaching probability in English today.

	Cournot 1843	*Schnuse 1849*	*Possible English*
7	épreuve aléatoire	Versuch	trial
4	événement aléatoire	aleatorisch Ereignis	event
4	spéculation aléatoire	aleatorische Spekulation	financial speculation
3	prime aléatoire	Prämie	risk premium
3	marché aléatoire	aleatorische Handel	financial market
2	condition aléatoire	Umstand	condition
2	droit aléatoire	eventuell Recht	contingent claim
2	instrument aléatoire	aleatorisch Instrument	chance device
1	convention aléatoire	aleatorische Uebereinkunft	bet
1	gain aléatoire	eventuell Gewinn	possible gain

Synonyms for *fortuit* in Cournot: *éventuel* and *accidentel*

Table 3: Nouns that Cournot used with *éventuel* and *accidentel* in his 1843 *Exposition* [78], with approximate counts of the number of sections where the terms appear, translations used by Schnuse in his 1849 German translation [79], and translations that might be used in teaching probability in English today.

	Cournot 1843	*Schnuse 1849*	*Possible English*
1	droit éventuel	eventuell Recht	contingent claim
1	gain éventuel	eventuell Gewinn	possible gain
1	perte éventuelle	eventuell Verlust	possible loss
3	cause accidentellle	unregelmässige Ursache	random cause
1	trouble accidentelle	zufällige Störung	random disturbance
1	circonstance accidentelle	—	random circumstance

Table 4: Probability books published in the first half of the twentieth century in languages other than Russian that followed Markov in using $X, Y, \ldots$ for probabilized quantities and $x, y, \ldots$ for their possible values. (This list is meant to be exhaustive; please let me know about any others that I have missed.)

	Name of quantity	*Name of mean*	*Operator*
1914	Emanuel Czuber in German [103], third edition of volume 1.		
	Größe	*Mittelwert*	$M(X)$
1919	Guido Castelnuovo in Italian [59].		
	variabile causale	*valore medio teorico*	$M(X)$
1925	Julian Coolidge in English [77].		
	variable	*mean value*	None
1925	Paul Lévy in French [241].		
	variable éventuelle	*valeur probable*	None
1928	Georges Darmois in French [106].		
	variable aléatoire	*valeur probable* *espérance mathématique*	$E(X)$

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	variable aléatoire	*valeur probable* *espérance mathématique*	$E(X)$
1937	Paul Lévy in French [244].		
	variable aléatoire	*valeur probable*	$\mathcal{M}\{X\}$
1937	Harald Cramér in English [89].		
	random variable	*mean value* *mathematical expectation*	$E(X)$
1949	Harald Cramér in Swedish [90]. (I have not yet seen the 1927 version of this book.)		
	tillfällig variabel *stokastisk variabel*	*medelvärdet*	$E\{X\}$
1950	William Feller in English [145].		
	random variable	*expectation* *mean* *mathematical expectation* *expected value* *average*	$\mathbf{E}(X)$

French: *Variable aléatoire* wins in 1928-1931.

Darmois had been teaching statistics in Paris for 5 years when he published his textbook using *variable aléatoire* in 1928.

In 1929, Fréchet returned to Paris to teach probability at the Ecole Normale, where he adopted Darmois's terminology as he realized that probability theory could be embedded in his earlier work in functional analysis. Thus a *variable aléatoire* became a real-valued function.

Khinchin quickly followed Fréchet in using *variable aléatoire* when he wrote in French and then also adopted *zufällige Variable* in German. By 1931 everyone, including Lévy, was following Fréchet and using *variable aléatoire* in French.

German: *stalemate before the war*

In the early 1930s, Khinchin, Slutskii, and Anderson were using *zufällige Variable*. Chuprov had championed it but had died in 1926.

Bortkiewicz, who had championed *zufällige Größe*, died in 1931, but Kolmogorov used it in his 1933 *Grundbegriffe* and elsewhere.

No native speakers of German had comparable stature in probability theory in the 1930s.

After the war, the logic of the mathematics and the language asserted itself with *Zufallsvariable*.

English: *random variable* wins in 1937-1940.

Harald Cramér's 1937 book very influential.

Jerzy Neyman's decision to use the *random variable* was key. It was dominant by the 1940s.

Joe Doob's *chance variable* was hopelessly in the minority, especially after the death of Abraham Wald in 1950.

Numbers of articles in five leading journals that used different terms for *random variable.*

	1930s	1940s	1950s	1960s
Annals of Mathematical Statistics				
chance variable	4	37	53	9
random variable	12	108	310	776
stochastic variable	2	11	3	8
statistical variable	2	4	0	0
random variate	0	2	6	4
Biometrika				
chance variable	3	1	2	4
random variable	2	23	83	222
stochastic variable	1	2	3	0
statistical variable	2	3	0	1
random variate	0	0	4	7
Journal of the American Statistical Association				
chance variable	0	6	8	4
random variable	1	13	94	284
stochastic variable	0	2	3	6
statistical variable	1	1	1	0
random variate	0	1	3	2
Journal of the Royal Statisical Society				
chance variable	1	2	0	0
random variable	4	10	18	59
stochastic variable	1	1	1	1
statistical variable	2	0	2	1
random variate	0	0	0	3
Transactions of the American Mathematical Society				
chance variable	6	8	2	0
random variable	0	5	40	72
stochastic variable	0	0	1	0
statistical variable	0	0	0	0
random variate	0	0	0	0

Counts from Google, 19 March 2016

“случайная переменная”	7,280
“случайная величина”	179,000
“Zufallsvariable”	103,000
“zufällige Variable”	3,270
“zufällige Größe”	3,390
"stochastische Variable“	3,290
“variable aléatoire”	172,000
“variable éventuelle”	952
“variable stochastique”	1,980
“random variable”	3,920,000
“chance variable”	11,600
“stochastic variable”	121,000

Question 4. Why do I care?

For 20 years, I have been working with Vladimir Vovk on the game-theoretic generalization of measure-theoretic probability.

- Probability and Finance: It's Only a Game!, Wiley, 2001.
- www.probabilityandfinance.com.
- "Lévy's zero-one law in game-theoretic probability", by Shafer, Vovk, and Takemura.

What words should we use for the generalizations of random variable and expectation in this new framework?

Generalizing Kolmogorov's axioms

Suppose $\mathcal{X}$ is a nonempty set let $\mathbb{G}_{\mathcal{X}}$ be the set of bounded-below extended real-valued functions on $\mathcal{X}$.

An *upper expectation* is a functional $\overline{\mathbf{E}} : \mathbb{G}_{\mathcal{X}} \to (-\infty, \infty]$ obeying these axioms.

Axiom E1. If $f_1, f_2 \in \mathbb{G}_{\mathcal{X}}$, then $\overline{\mathbf{E}}(f_1 + f_2) \leq \overline{\mathbf{E}}(f_1) + \overline{\mathbf{E}}(f_2)$.

Axiom E2. If $f \in \mathbb{G}_{\mathcal{X}}$ and $c \in [0, \infty)$, then $\overline{\mathbf{E}}(cf) = c\overline{\mathbf{E}}(f)$.

Axiom E3. If $f_1, f_2 \in \mathbb{G}_{\mathcal{X}}$ and $f_1 \leq f_2$, then $\overline{\mathbf{E}}(f_1) \leq \overline{\mathbf{E}}(f_2)$.

Axiom E4. For each $c \in \mathbb{R}$, $\overline{\mathbf{E}}(c) = c$.

We interpret $\overline{\mathbf{E}}(f)$ as the lowest price at which you can buy f.

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The game

Skeptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots$:
  Forecaster announces an upper expectation $\overline{\mathbf{E}}_n$ on $\mathcal{X}$.
  Skeptic announces $f_n \in \mathbb{G}_{\mathcal{X}}$ such that $\overline{\mathbf{E}}_n(f_n) \leq \mathcal{K}_{n-1}$.
  Reality announces $x_n \in \mathcal{X}$.
  $\mathcal{K}_n := f_n(x_n)$.

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For each bounded-below extended real-valued function X on the path $\overline{\mathbf{E}}_1, x_1, \overline{\mathbf{E}}_2, x_2, \ldots$, set

$$\overline{\mathbb{E}}(X) := \inf\{\mathcal{K}_0 \mid \text{Skeptic can guarantee} \liminf K_n \geq X\}.$$

Theorem. $\overline{\mathbb{E}}$ is an upper expectation.

- What should we call f and X? Gambles? Variables? Random variables?
- What should we call $\overline{\mathbf{E}}$ and $\overline{\mathbb{E}}$? Upper previsions? Upper expectations? Upper prices?
- What should we call $\overline{\mathbf{E}}(f)$ and $\overline{\mathbb{E}}(X)$? Upper expected values? Upper prices?