Reviving Pascal's and Huygens's Game-Theoretic Foundation for Probability

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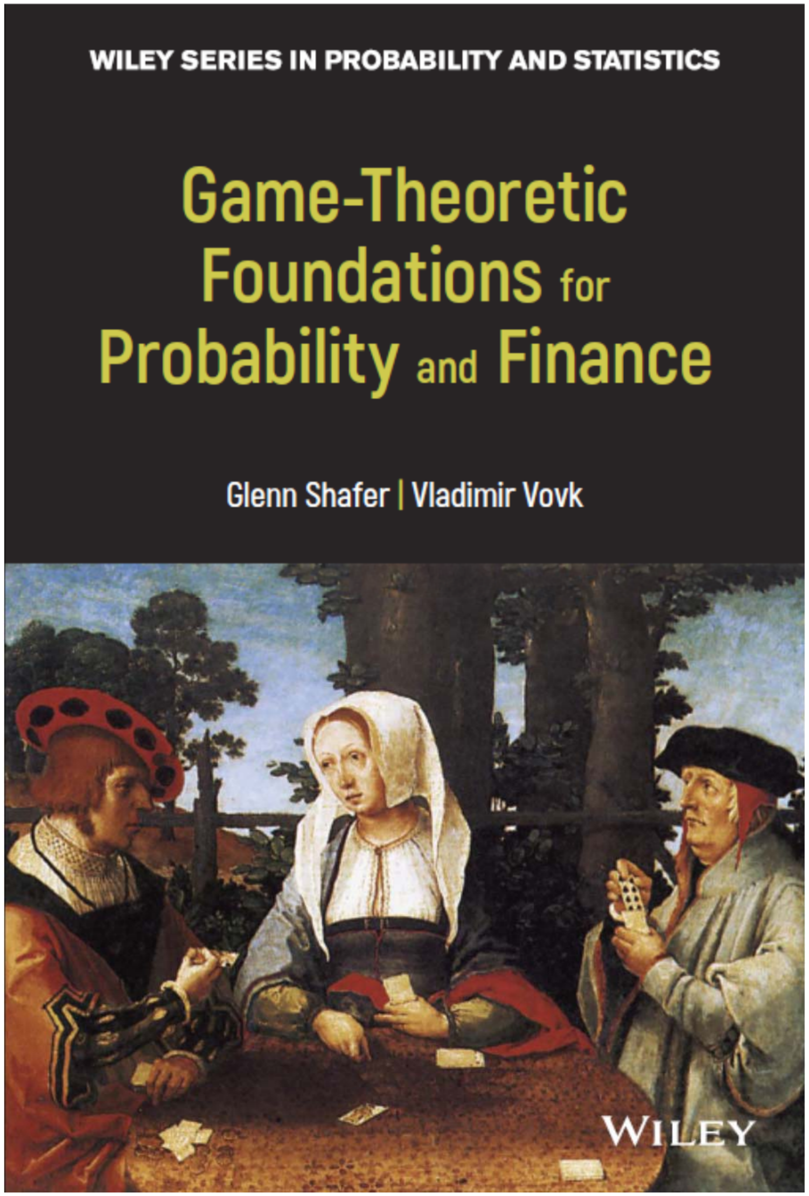
Department of Philosophy, University of Utrecht, December 19, 2018

Pascal and Huygens based the calculus of chances on the structure of games, not on frequency.

This idea quickly disappeared, because the picture of equally frequent cases was so entrenched.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Glenn Shafer | Vladimir Vovk
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ON THE BACK COVER

Ever since Kolmogorov's *Grundbegriffe*, the standard mathematical treatment of probability theory has been measure-theoretic. In this ground-breaking work, Shafer and Vovk give a game-theoretic foundation instead. While being just as rigorous, the game-theoretic approach allows for vast and useful generalizations of classical measure-theoretic results, while also giving rise to new, radical ideas for prediction, statistics and mathematical finance without stochastic assumptions. The authors set out their theory in great detail, resulting in what is definitely one of the most important books on the foundations of probability to have appeared in the last few decades.

– Peter Grünwald, CWI and the University of Leiden

Two foundations for probability:

- Measure theory
- Game theory

The game-theoretic foundation goes deeper:

- Probabilities are derived from a perfect-information game.
- To prove a theorem, you construct a strategy in the game.

Classical picture (Fermat and dice players over the millenia):

Equally frequent chances.

Modern version (pure mathematics):

Probability measure.

- Space of outcomes.
- Measure with total measure one.

Connect probability with frequency by Bernoulli's theorem (unless event of very small probability happens).

Pascal and Huygens and the commercial arithmetics:

Players treated equally.

Modern version (pure mathematics):

Game with three players:

- Forecaster offers odds.
- Skeptic decides how to bet.
- Reality decides the outcomes.

Connect probability with frequency by Bernoulli's theorem (unless Skeptic multiplies the capital he risks by large factor).

1. The calculus of chances before Pascal and Fermat
2. The division problem
3. Pascal’s game theory
4. Huygens’s game theory
5. Back to frequency
6. Modern game theory

Laplace launched the legend of Pascal and Fermat in 1795:

"Probability theory owes its birth to two French geometers of the 17th century."

Patriotic words!

Blaise Pascal
1623-1662

Pierre Fermat
1607-1665

Laplace was more careful in 1814:

For quite a long time, people have ascertained the ratios of favorable to unfavorable chances in the simplest games; stakes and gambles were fixed by these ratios.

But before Pascal and Fermat, no one gave principles and methods for reducing the matter to calculation, and no one had solved problems of this type that were even a little complicated.

So we should attribute to these two great geometers the first elements of the science of probabilities…

Laplace’s less nuanced statement is often echoed:

Lacroix, 1816:

The probability calculus, invented by Pascal and Fermat, has never since ceased exciting the interest and exercising the wisdom of their most illustrious successors…

Poisson, 1837:

A problem about games of chance proposed to an austere Jansenist by a man of the world was the origin of the calculus of probabilities.

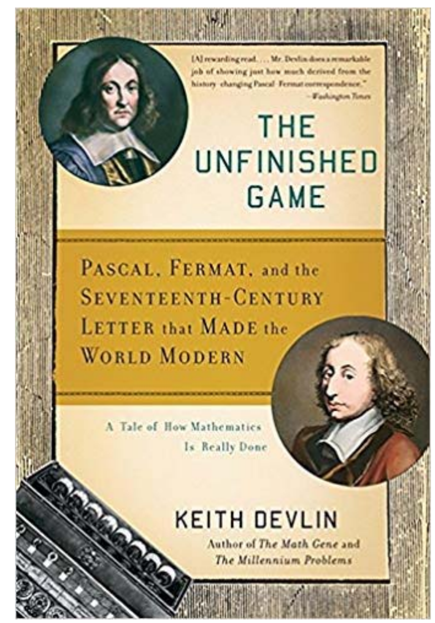
Conceptual revolution?

Ian Hacking, born 1936

Keith Devlin, born 1947

Ian Hacking, 1976: Probability, as we now conceive it, came into being about 1660. It was essentially dual, on the one hand having to do with degrees of belief, on the other, with devices tending to produce stable long-run frequencies.

Keith Devlin, 2008: The Pascal-Fermat correspondence showed that it is possible to use mathematics to see into the future.



Before Pascal and Fermat

Laplace was right:

Long before Pascal and Fermat, people

- ascertained ratios of favorable to unfavorable chances and
- fixed stakes and bets using these ratios.

HOW?

Long before Pascal and Fermat, people

- ascertained the ratios of favorable to unfavorable chances, **(multiply)**
- fixed stakes and bets using these ratios. **(rule of three)**

Latin poem *De Vetula*, written in Paris about 1250.

- Counted the 216 chances for three dice.
- Each has the same force and frequency.

Sum of points	3	4	5	6	7	8	9	10	Total
# of chances	1	3	6	10	15	21	25	27	108

Sum of points	18	17	16	15	14	13	12	11	Total
# of chances	1	3	6	10	15	21	25	27	108

David Bellhouse
Born 1948

David Bellhouse called *De Vetula* a **"medieval bestseller"**.

- Nearly 60 copies of manuscript survive.
- Editors of the printed versions of 1479, 1534, and 1662 understood the counts.

Long before Pascal and Fermat, people

- ascertained the ratios of favorable to unfavorable chances,
- fixed stakes and bets using these ratios.

De Vetula counted the chances for the 16 possible sums.

Sum of points	3	4	5	6	7	8	9	10	Total
	18	17	16	15	14	13	12	11	
# of chances	1	3	6	10	15	21	25	27	216
But *De Vetula* did **not** turn the counts into probabilities.									
Probabilities	1/216	3/216	6/216	10/216	15/216	21/216	25/216	27/216	
	0%	1%	3%	5%	7%	10%	12%	12%	

Without probabilities, how do you fix stakes and bets?

RULE OF THREE

RULE OF THREE

You buy 15 bushels of wheat for 10 shillings. What do you charge for 3 bushels? Answer: 2 shillings.

This is the rule of three: find the 4^{th} number in a proportion from 3 that are known.

For us, this is a matter of algebra: $15/3 = 10/x$, and so $x = 2$ shillings.

But al-Khwarizmi's 9^{th}-century algebra was all in words.

Medieval commercial arithmetics used the rule of three in problem after problem:

- trading in goods,
- dividing profits,
- changing currencies,
- pricing alloys, etc., etc.

Occasionally, for fun, an author might throw in a problem about a game.

Algebra with symbols emerged only in the Renaissance, developed by the authors of commercial arithmetics: Italian abbacus masters, German reckoning masters.

Sum of points	3	4	5	6	7	8	9	10	Total
	18	17	16	15	14	13	12	11	
# of chances	1	3	6	10	15	21	25	27	216

Question:

Three dice are thrown repeatedly. **Player A** bets on 9; **Player B** bets on 15.
In other words,

- **Player A** gets the money on the table if 9 comes up before 15.
- **Player B** gets the money on the table if 15 comes up before 9.

Player A puts 5 pistoles on the table.
How much should **Player B** put on the table?

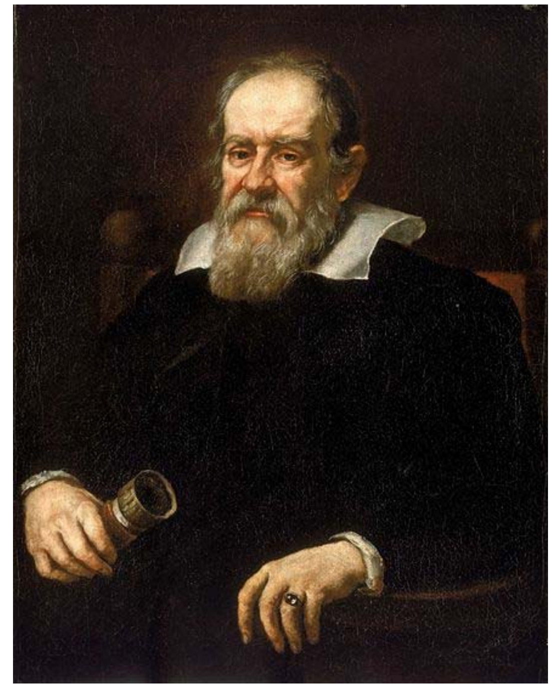
Solution:

1. The ratio of 9's chances to 12's chances is 25 to 10.
2. So **Player A** wins only 25 times for every 10 times **Player B** wins.
3. **Player A** put 5 pistoles on the table for **Player B** to win.
4. **Player B** puts x on the table for **Player A** to win.
5. So **B** will win 10 x 5 pistoles every time **A** wins 25 x x.
6. To make this fair, set x = 2 pistoles: 10 x 5 = 25 x 2.

RULE OF THREE

Scholars who wrote about counting chances before Pascal and Fermat.

Geralomo Cardano
1501 - 1576



Galileo Galilei
1564 - 1642

Pascal

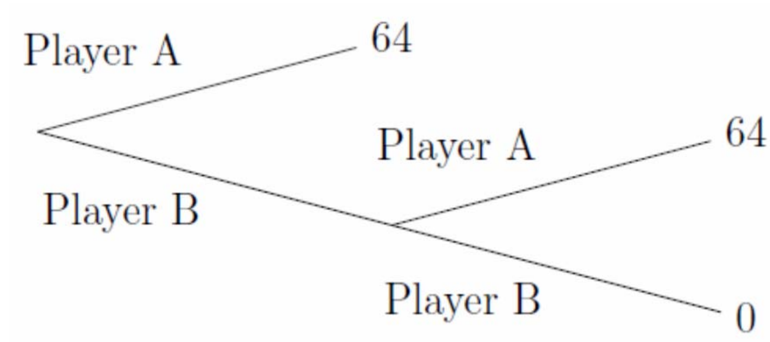
Pascal's research program at the beginning of 1654 included a new field of research:

What we call in French *faire les partys des jeux*, where the uncertainty of fate is so well overcome by the rigor of calculation that each of two players can see themselves assigned exactly what they have coming.

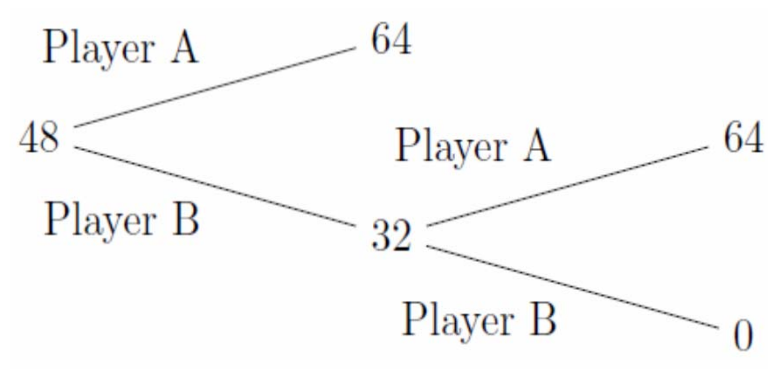
Has remained unsettled, because it can only be found by reasoning, not by experience.

Treatise will have the surprising title *Geometry of Chance*.

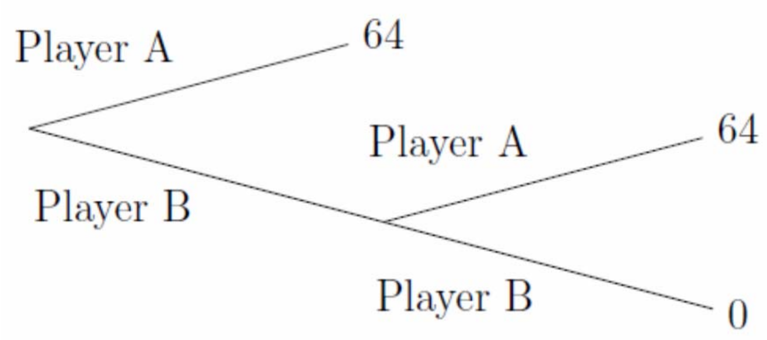
Pascal’s division problem



Pascal’s solution



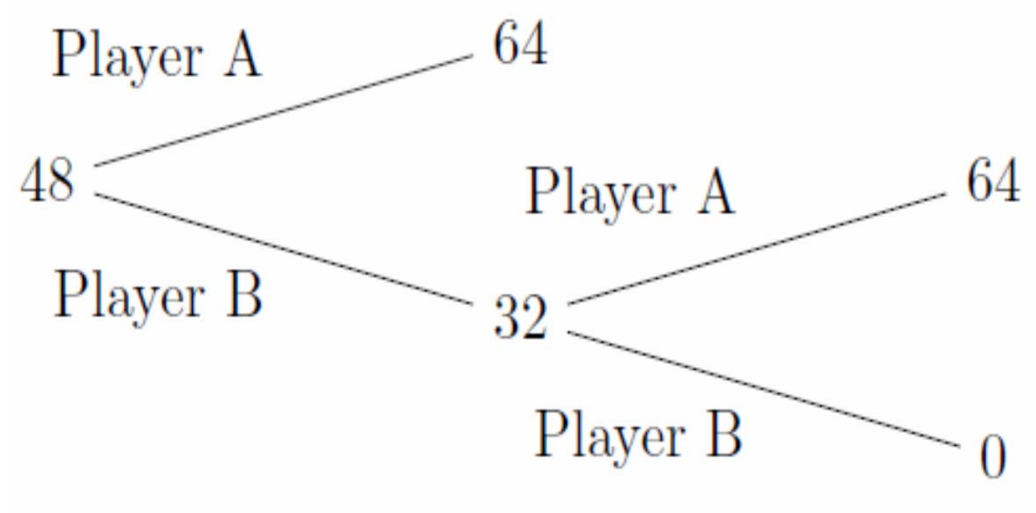
Pascal's division problem



Occasionally discussed in commercial arithmetics.

Discussed in print by Pacioli, Cardano and Tartaglio.
They gave various solution using the rule of three.

Never a game of chance!

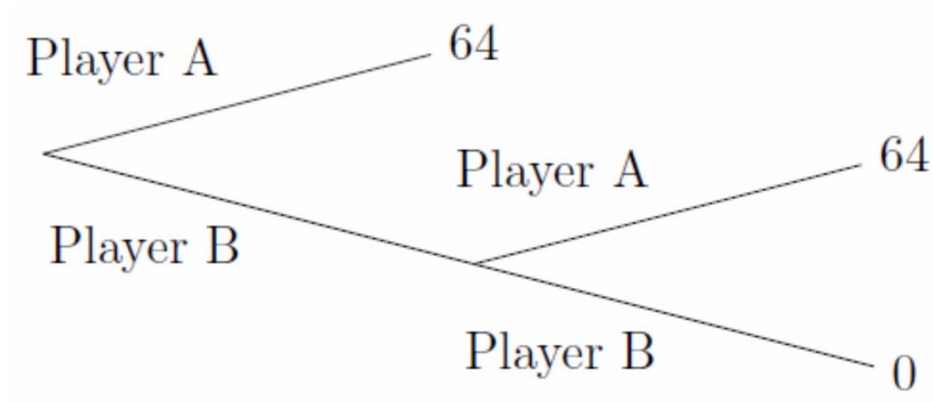


Pascal's two principles:

1. Take any amount you are sure to get.

2. *If the game is one of pure chance*, and the chances for winning a certain amount are equal, then divide the amount equally.

Michael Mahoney
1939 – 2008
Fermat's biographer



Fermat's solution: There are four equal chances:

Player A wins the first round, Player A wins the second round;
Player A wins the first round, Player B wins the second round;
Player B wins the first round, Player A wins the second round;
Player B wins the first round, Player B wins the second round.

Player A wins in three our the four chances and so should get ¾ of the stakes.

Pascal's response: My solution is better because **it carries its demonstration in itself**.
(Your solution requires experience of frequencies.)

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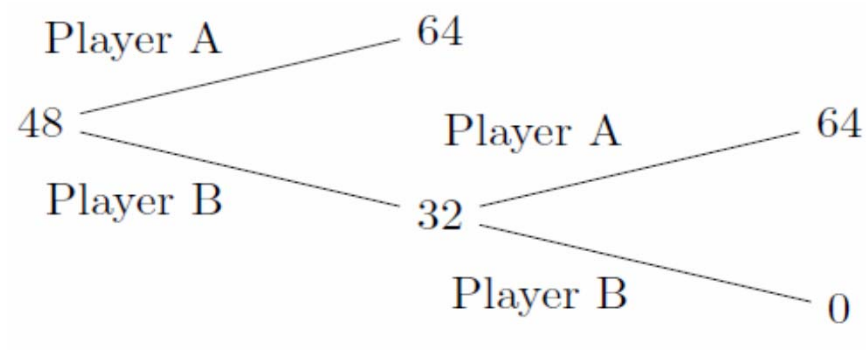
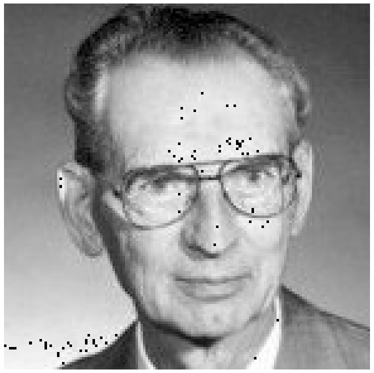


In the *weltlich* tradition, always a problem about a ball, archery, or chess tournament.

Why assume the game is one of pure chance?

In 2003, researchers found a commercial arithmetic in the Vatican library, dating from about 1400, that made Pascal's argument without the pure-chance assumption.



Anders Hald:
"The division problem was first solved by Pascal and Fermat in 1654..."

Hald was wrong!

Huygens

Pascal's letters to Fermat.

- Written in 1654.
- Published in 1679.

Pascal's *Triangle arithmétique*.

- Written at the end of 1654.
- Published in 1665.
- Rare.

Blaise Pascal
1623 - 1662

Huygens's *De Ratiociniis in ludo aleae*.

- Inspired by 1655 visit to Paris.
- Drafted 1656.
- Published 1657.
- Widely distributed and translated.

Christiaan Huygens
1629 - 1695

Having learned about the Pascal-Fermat correspondence from mathematicians in Paris, Huygens saw an opportunity to use Descartes's algebra.

René Descartes
1596 - 1650

Use equations to express conditions on a number x.

- *Analysis:* Find what x must be if there is a number satisfying the conditions.
- *Synthesis:* Prove that the number found does satisfy the conditions.

Proposition I. If I have the same chance to get *a* or *b* it is worth as much to me as $(a + b)/2$.

Consider this fair game:

- We both stake *x*.
- The winner will give *a* to the loser.

The analysis:

- If I win, I get $2x - a$.
- If this is equal to b, then $x = (a + b)/2$.

The synthesis:

- Having $(a + b)/2$, I can play with an opponent who stakes the same amount, on the understanding that the winner gives the loser *a*.
- This gives me equal chances of getting *a* or *b*.

Pascal proved the same thing using his two principles and the assumption that the game is one of *pure chance*.

Huygens's argument is better:

- Clearly does not require that the game be one of pure chance.
- Replaces principle of equal division by willingness of players to play on even terms.

Proposition I. If I have the same chance to get a or b it is worth as much to me as $(a + b)/2$.

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- This gives me equal chances of getting a or b.

Hans Freudenthal
1905 – 1990

Emphasized the difference between Huygens's argument and the modern definition of "expectation".

Proposition III. If I have p chances for a and q chances for b, this is worth $(pa + qb)/(p + q)$.

Synthetic (constructive) proof:

- Assign each chance to a different player.
- I am one of the $p + q$ players.
- Each of us puts up $(pa + qb)/(p + q)$.
- Winner takes all.

- I make side bet with q opponents; winner gives loser b.
- I make side bet with other $p – 1$ opponents; winner gives loser a.

- This gives me p chances for a and q chances for b.

$$(p + q)\ (pa + qb)/(p + q) – qb – (p – 1)a = a$$

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Ivo Schneider
Born 1938

Ivo's objection to Huygens:
Not all players are treated the same if one gets to decide what side bets to make.

At the end of *De rationciniis*, Huygens stated 5 problems with answers but no solutions.

- Fermat had proposed Problems 1 and 3.
- Pascal had proposed Problem 5.

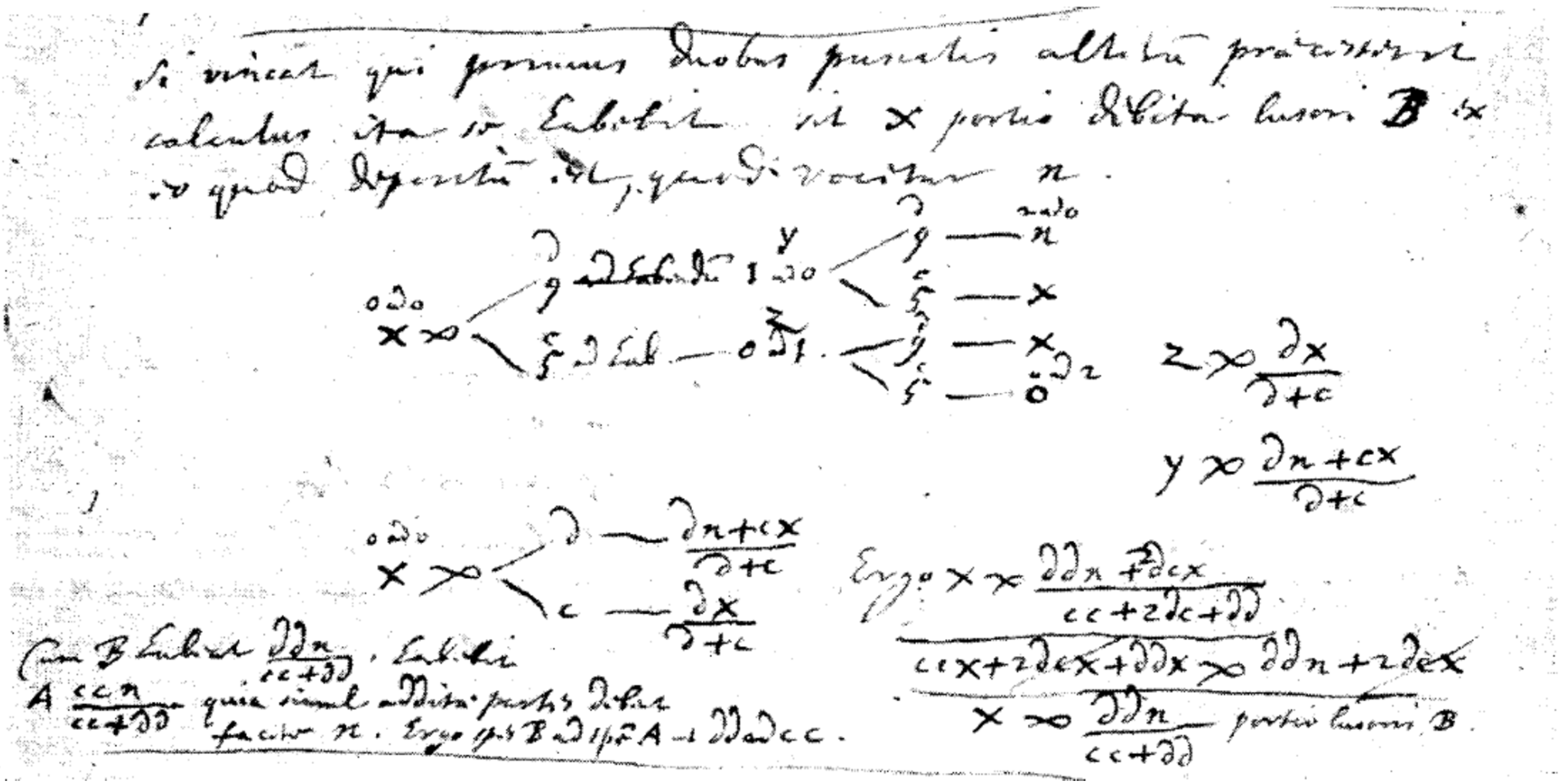
We can justify Laplace's words with the claim that probability theory was launched by these problems.

Huygens's 5th problem:

- Having each taken 12 coins, A and B play with 3 dice.
- A gives a coin to B each time he gets an 11.
- B gives a coin to A each time he gets a 14.
- The winner be the one who first has all the coins.

The ratio of A's chance to B's chance is 244140625 to 282429536481.

The first surviving document showing with an event tree is in a manuscript dated August 1676, in which Huygens solves Problem 5.



Back to Frequency

Pascal’s and Huygens’s game-theoretic foundations were quickly pushed aside by the deeply entrenched concept of equally frequent chances.

Pierre Rémond de Monmort
Essay d’analyse sur les jeux de hazard 1708

Abraham De Moivre
De mensura sortis 1711

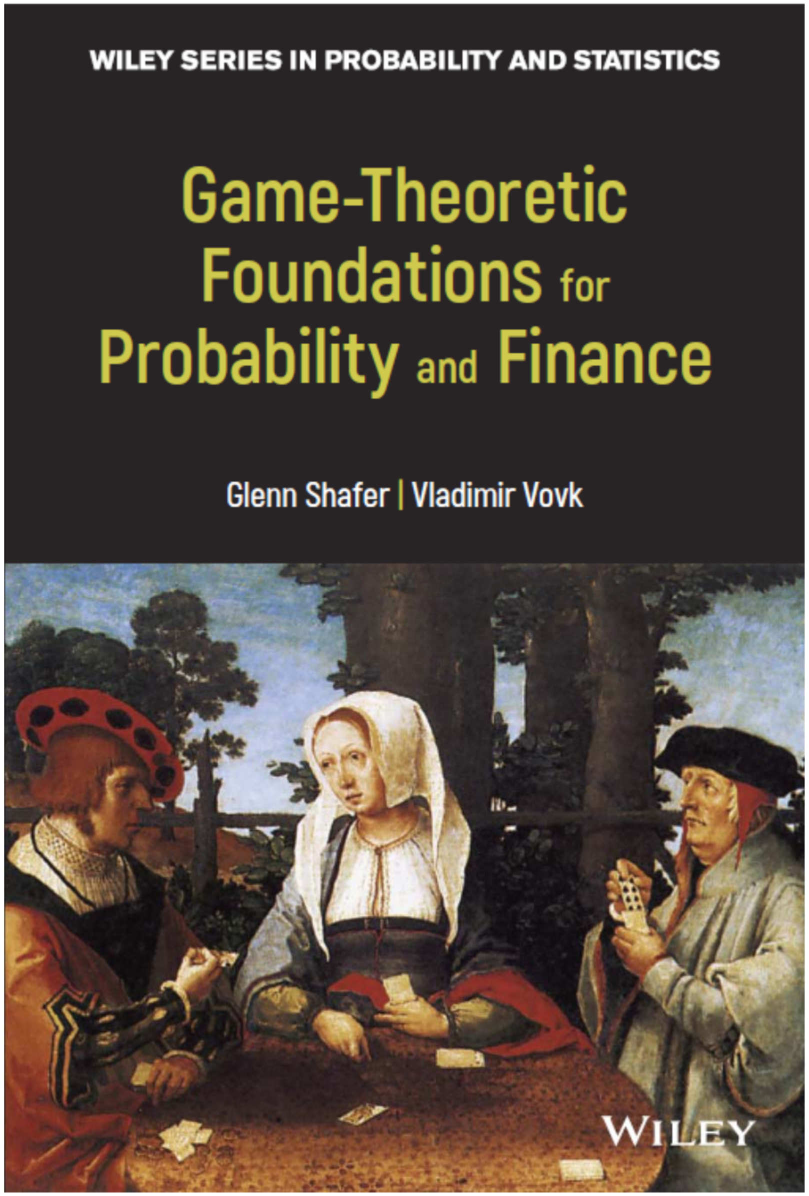
Jacob Bernoulli
Ars conjectandi 1713

Conclusion

Pascal and Fermat provided a game-theoretic foundation for probability theory.

But it did not satisfy the standards of rigor now demanded by game theory.

- What are the rules of play?
- As Ivo Schneider asks, why does one player get to make the side bets he wants?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Fermat and the dice players: Equally frequent chances.

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Obtain frequencies by Bernoulli's theorem (unless event of very small probability happens).

Pascal and Huygens and the commercial arithmetics: Players treated equally.

Make it modern, pure mathematics:

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