The Notion of Event in Probability and Causality

Situating myself relative to Bruno de Finetti

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Abstract

As Bruno de Finetti taught us, the notion of event in a theory of probability is fundamental, perhaps determinative. In this paper, I compare the notion of event in de Finetti's subjective theory of probability with the more situated notion of event that underlies the theory of probability and causality that I have developed over the past ten years.

Outline

Acknowledgements

This paper responds to Paolo Garbolino's suggestion that I compare de Finetti's notion of event with my own. I am grateful to Professor Garbolino for the invitation to present this comparison at the University of Pisa and to Domenico Costantini and Maria Carla Galavotti for their invitation to present it at the University of Bologna. I would also like to acknowledge the support of the German Fulbright Commission, which sponsored my stay at the Free University of Berlin, where this paper was prepared, as well as the hospitality of my host at the Free University, Professor Hans-J. Lens.

1. Introduction

As Bruno de Finetti taught us, the notion of event in a theory of probability is fundamental, perhaps determinative. This paper grows out of Paolo Garbolino's suggestion that I compare the notion of event in de Finetti's subjective theory of probability with the somewhat different notion of event in the theory of objective probability and causality that I have been developing over the past ten years.

The paper has become somewhat autobiographical. Perhaps this was inevitable, for de Finetti has been a constant stimulus during my intellectual career. In the late 1960s, when I first began to study probability and mathematical statistics, de Finetti's interpretation of probability as subjective price was already as an inescapable challenge to anyone with philosophical ideas of his own about probability. It had to be addressed not only because of its power and self-consistency, but also because of its claim of exclusivity. In the foundational debates of the 1930s, de Finetti had advocated his subjective interpretation not as one of many ways of understanding probability but as the correct way of understanding it, and this attitude was inherited by the Bayesian statisticians of the 1970s, who saw incoherence not only in the thinking of their traditional frequentist opponents but also in newer heresies. The heresy that I promoted in the 1970s and 1980s, Dempster-Shafer belief functions (Shafer 1976), was an alternative theory of subjective probability, not a theory of objective probability like that of the frequentists, and this made my Bayesian colleagues even less patient with it.

My difference of opinion with the Bayesians centered on Dempster's rule for combining belief functions. This rule combines degrees of belief (numerical degrees of belief—like probabilities but perhaps not additive) based on different bodies of evidence to obtain degrees of belief based on the pooled evidence. This generalizes the idea of combining initial subjective probabilities with new evidence by Bayes's rule, and so it challenges the Bayesian view that all changes in belief should proceed by Bayes's rule, or at least by Bayesian conditioning: new evidence B should always result in the subjective probability P(A) being replaced by the conditional probability $P_B(A)$. My Bayesian colleagues regularly counseled that I should read de Finetti to learn the error of my ways.

Remarkably enough, de Finetti's himself did not subscribe to the dogma, so widely and loudly held by my Bayesian colleagues, that changes in belief should always proceed by Bayesian conditioning. For him, the revision of belief was always a matter of fresh reflection and self-examination, never a matter of rote application of a mathematical rule. There is, in fact, nothing in de Finetti's writing that justifies the claim that changing probabilities other than by conditioning is incoherent. So why did my Bayesian colleagues think there is? How did they make their leap from what de Finetti actually said?

Under some circumstances, the Bayesians are right. If the probabilities with which we begin include probabilities for A and for B, and if we know in advance that if B is true we will soon learn it is true (and nothing more), then coherence (avoiding losing money to a clever opponent when our probabilities are interpreted as offers to bet) does require that we change our probability for A from to the conditional probability $P_B(A)$. But in my opinion (and apparently also in de Finetti's opinion), this condition is seldom met. So I soon formed a plan for persuading the Bayesians that they were mistaken. I would explain the conditions for the validity of Bayesian conditioning more clearly than anyone else had done, so clearly that the Bayesians themselves would see that these conditions are seldom met—so clearly that they would see that they were in the same boat with their rule of conditioning as I was with my rule of combination. Then they would improve their manners and be better boatmates.

Things did not quite work out as I planned. Few of my Bayesian colleagues saw the light, and I finally tired of the argument. But my enterprise of understanding the conditions for conditioning took on a life of its own. Gradually I realized that I was proposing a new foundation for probability in general—especially for objective probability and probabilistic causality.

The contrast between the Finetti's theory of subjective probability and my own new theory of objective probability and causality begins with and is largely encapsulated by the different notions of event with which they work. For de Finetti, an event is timeless; it is better described as something that is true or false rather than as something that happens. In my theory of causality, in contrast, events happen. They are situated in a dynamic structure of possibility, which unfolds into reality in time and space.

In the next section of the paper, I review de Finetti's timeless notion of event, with some attention to its origin, which I locate in the measure-theoretic framework that was triumphing in mathematical probability as de Finetti entered the arena. In the next two sections, I explain why I believe, in the end, that this measure-theoretic notion of event is inadequate even for a subjective theory of probability, in the proper sense of the term. Finally, in a concluding section, I sketch the more structured notion of event that I see at the base of an adequate concept of objective probability.

2. De Finetti's Timeless Notion of Event

An *event*, by etymology and use, is something that happens, at some particular time and place. As such, it can be contrasted with a *fact*, which does not happen but is merely true. Yes, there is an etymological trace of happening in *fact*, for it derives from the past participle of *facere*, but this is merely etymology. A fact (the speed of light, for example) need not be the result of anyone's action or anything's happening, even in the remote past. I make this point in English, but we find the same contrast in other European languages: *evènement* versus *fait* in French, *Ereignis* versus *Tatsache* in German, *evento* versus *fatto* in Italian.

Quite remarkably, de Finetti turned the ordinary distinction between fact and event on its head. This is spelled out sharply, for example, in his 1979 course on the philosophy of probability, where he tells us to distinguish between a fact that can happen and an event that can be verified (*«fatto» che può «accadere»* and *«evento» che può «verificarsi»*, de

Finetti 1995, p. 235). In "La nozione di evento" (1952), he spells out his notion of event in this way:

> …an event is a logical entity which can assume the two values, true or false (that is: it did occur or it will happen, respectively, it did not occur or it will not happen). In a specific state of information (for an individual, or for the collectivity) an event is certain, or impossible (when its result is known and it is respectively true or false), or possible (when its result is not known). (translation by Mara Khale and Antonella Ansani, Finetti 1993, p. 416)

Here time enters only to be immediately suppressed. Yes, an event happens or fails. Yes, there may be a time of its happening. But this is irrelevant.

Irrelevant to what? Irrelevant to probability. When an event happens may be of practical importance for many purposes, but according to de Finetti, it is irrelevant to the definition and meaning of probability:

> "no distinction related to the nature of the considered events is intrinsically relevant to the definition of probability (even if, as is obvious and as we will say, it may have practical influence on the methods of probability evaluation)".
>
> For instance, it is irrelevant to differentiate between past or future events: the probability of a future event is evaluated because its result is unknown by us, in the same way as we do not know the result of many past events, and of course not because it is indeterminate… (ibid., p. 417)

This timeless notion of event buttresses supports, of course, de Finetti's insistence on a subjective conception of probability. There is no room for probability as an objective attribute of an event, because, as he says, "every objective meaning of the event disappears in its being true or false..." (idid., p. 416).

De Finetti's insistence on a flat and timeless meaning for *event* appears to derive from his ambition for his subjective conception of probability. Had he been content with making subjective probability merely one special use for the mathematics of probability, he might have assigned subjective probabilities to *propositions* rather than to *events*. But since the early eighteenth century, mathematical probability has been concerned with the probabilities of *events*, and only by giving a timeless meaning to *event* could de Finetti make plausible for mathematical statisticians his message that all useful and meaningful probabilities are subjective.

We must recognize, however, that de Finetti did not originate the timeless notion of event for probability. Nor was he responsible for its nearly universal adoption in the early twentieth century. Much more influential were those, such as Maurice Fréchet, Francesco Cantelli, Hugo Steinhaus, and most notably Andrei Kolmogorov, who were motivated not by a subjectivist and anti-realist philosophy but by a quest for mathematical clarity. Their way of making probability mathematically clear and hence respectable, expressed definitively in Kolmogorov's *Grundbegriffe der Wahrscheinlichkeitsrechnung* (1933), was to equate it with a measure on a sample space. In this formulation, the sample space is comprised by the possible outcomes of an experiment, and an *event* is a subset of this sample space. This formulation is broad enough to encompass stochastic processes, which were being intensely studied at the time, by Kolmogorov and de Finetti among others, but in its abstraction it ignores any structure of time or change.

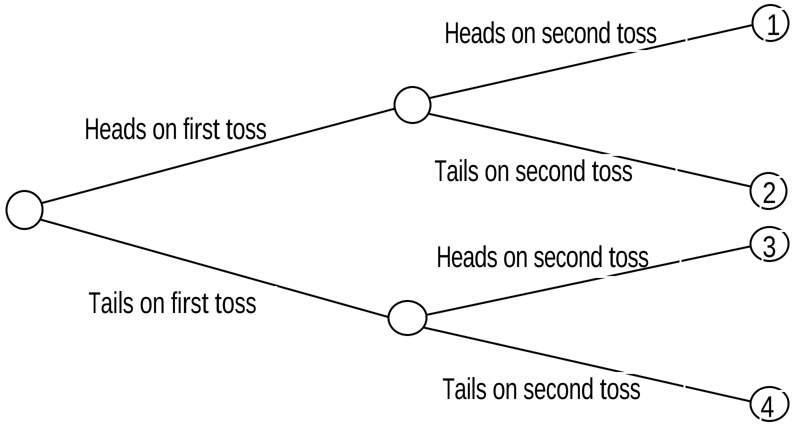

Figure 1. A time structure reduced to a set of four points.

In case these last words sound mysterious, let me discuss the simple example in Figure 1. This figure shows two successive tosses of a coin, but the abstract theory represents it in a way that suppresses this time ordering. The sample space consists of the four possible paths that events might take; Ù = {1,2,3,4}. This is a sample space like any other; each subset of {1,2,3,4} is an event like any other. When we speak of a probability measure on Ù, we are working at a level of abstraction that ignores any structure that goes beyond the relation between Ù and its elements. In particular we are ignoring the time, which now lies hidden inside the four points. When an event that happen as the coins are tossed is reduced in this way to a subset of Ù, its situatedness in time disappears. As de Finetti puts, it this situatedness disappears in the event's "being true or false". We no longer see whether the event necessarily happens or fails on the first toss (like the event {1,2}), necessarily happens or fails on the second toss (like the event {1,3}), or might happen on either (like {1,2,3}).

De Finetti formulated his philosophy of probability at a time when the measure-theoretic mathematical foundation for probability carried all the self-evidence of an exciting new idea. It is for this reason, perhaps, that the irrelevance of deeper structure for the meaning of an event was so obvious to him—so obvious that he could use it as a self-evident starting point to argue for the subjective meaning for probability.

3. The Two Roles of Event: Bearer of Probability and Giver of Probability

When we compare the measure-theoretic framework for probability that emerged in the early twentieth century with the less abstract and less sharply defined frameworks that were used in the eighteenth and nineteenth centuries, the notion of conditional probability emerges as one of the most striking novelties of the measure-theoretic framework. From the very beginning of mathematical probability, it was understood that the probability of a thing could change: it could be different under different circumstances, at different times, or under different hypotheses. There are many different words here, and perhaps many different concepts; it is not at all clear that they are to be formalized in a uniform way. The measure-theoretic framework brings a surprisingly uniform formalization: both the thing that has a probability and the circumstance or hypothesis that determines this probability is an event—a subset of the sample space. Moreover, any event seems suddenly to have the right to determine probabilities for other events; given any event B (at least any event B that itself has nonzero probability), we are authorized to speak of the *conditional probability* of another event A given B.

Twenty years ago, when I first began to puzzle over the notion of conditional probability, I tried to trace it back into the nineteenth century. Who, I asked, first introduced the notation $P_B(A)$ that Kolmogorov used in his *Grundbegriffe*? Who first introduced any notation that implies that the thing that determines the probability (here B) is an object of the same kind as the thing that has the probability (here A). And when did this identity between bearers and givers of probability start to seem obvious to mathematicians? My search turned up two pioneers: the American philosopher Charles Sanders Peirce, and the German mathematician Felix Hausdorff (see Shafer 1982). Peirce talked about the probability of one proposition given another, using a notation that did not catch on. Peirce's thoughts were echoed by some English logicians, but they seem not to have been noticed by mathematicians. It was left to Hausdorff, in a short article published in a short article published in 1900, to introduce (1) the idea that both the bearers and givers of probability are events—subsets of a sample space—and (2) the notation $P_B(A)$. Hausdorff's tone is revealing; he seems embarrassed to present as his invention something that seems so obvious and natural.

Making the givers of probabilities events just like the bearers of probability was an essential step in putting probability into the framework of measure theory. Kolmogorov's measure-theoretic framework begins with a probability measure P on the sample space, which gives a probability P(A) for every event A. The axioms of the theory, which are the same as the axioms for measure, are about these unconditional probabilities. Conditional probabilities are mathematically secondary; introduced by a definition rather than being axiomatized: the conditional probability $P_B(A)$ is defined, whenever $P(B) > 0$, by $P_B(A) := P(A\ \ B)/P(B)$.

The proponents of the new measure-theoretic framework—mathematicians such as Maurice Fréchet, Harald Cramér, William Feller, Joseph Doob, and Kolmogorov himself—all subscribed to objective conceptions of probability. Roughly speaking, they were frequentists. De Finetti was a major opponent of both the objective conception and the measure-theoretic framework; he insisted that the mathematical understanding of probability should start with the idea of subjective price rather than with the idea of measure. If one wanted axioms, these should be axioms for price, not axioms for measure. This opposition obscured the extent to which de Finetti actually accepted much of the new framework, and the extent to which he even relied on its authority to support his own philosophical positions.

I have already made this point with respect to the timelessness of the notion of event. Because the timeless concept of event (subset of a sample space) had become an accepted part of the mathematics of probability, de Finetti could take this timelessness as a starting point for arguing for a subjective conception of probability. I now want to make a similar point about conditional probability. De Finetti did not accept P(A B)/P(B) as the <u>definition</u> of the conditional probability $P_B(A)$.. But he started, without quibble, with an assumption that originated with the measure-theorists: the giver B and bearer A of a conditional probability $P_B(A)$ are both timeless events—subsets of a sample space. Here is his definition of a person's subjective conditional probability $P_B(A)$: it is the price the person assigns to a contract that pays $1 if A and B both happens, pays $0 if B happens but A does not, and is cancelled (the payment is returned) if B does not happen. This definition applies even if P(B) = 0, but if P(B) > 0, then the person must make $P_B(A)$ equal to P(A B)/P(B) if he wants to avoid incoherence—i.e., if he wants to avoid setting prices that would enable an opponent to make money from him for certain.

With his concept of conditional probability, de Finetti followed the measure-theorists into a formalism that leaves aside classical questions about how probabilities should change and, more generally, how they should depend on circumstances or information. Conditional probabilities $P_B(A)$, for de Finetti as for the measure-theorists, are defined or assessed in the same circumstances or with the same information as the unconditional probabilities P(A) are defined or assessed. If I make assessments P(A) and $P_B(A)$, and then I learn that B is true, should I change my probability for A from P(A) to $P_B(A)$? That depends, de Finetti tells us, and what it depends on is not treated by the formal theory. Why not? Is there any justification for leaving the question of how subjective probabilities should change outside the mathematical probability, aside from the fact that the now official measure-theoretic foundation does no better?

4. Timeless Events are not Adequate for Probability

Classically, starting with Pascal and Huygens and continuing with De Moivre and Laplace, probability theory did have a lot to say about how probabilities should change as events unfold and information is obtained. De Finetti was correct to conclude that the new measure-theoretical mathematical framework had eliminated this: once we make B flat and timeless, we can no longer say about how P(A) should change when B happens. But I think he and the measure-theorists were wrong to accept this new understanding of *event*. It did not and does not do justice to the classical content of the theory of probability.

The need for a dynamic rather than a flat concept of event for probability theory first became clear to me in the late 1970s, as I pondered some of the puzzles that turn on the danger in always identifying $P_B(A)$ as the probability that one should adopt when one learns B. There are many of these puzzles, although they seem to have appeared only in the twentieth-century, after the establishment of the measure-theoretic framework. My favorite involves an opponent holding two cards selected at random from a deck of four cards: A♥, A♠, 2♥, and 2♠. There are six equally likely hands:

Ù := { {A♥,A♠}, {A♥,2♥}, {A♥,2♠}, {A♠,2♥}, {A♠,2♠}, {2♥,2♠} }.

The probability that the opponent holds at least one ace is 5/6:

P(B) = 5/6, where B := { {A♥,A♠}, {A♥,2♥}, {A♥,2♠}, {A♠,2♥}, {A♠,2♠} }.
The probability that he holds both aces is 1/6:

P(A) = 1/6, where A := { {A♥,A♠} }.

So the conditional probability $P_B(A)$ is

$$P_B(A) = P(A \quad B)/P(B) = (1/6)/(5/6) = 1/5.$$

Suppose 5/6 and 1/6 are indeed my subjective probabilities for B and A, respectively. Suppose the opponent now informs me that B is true: "I have at least one ace." Suppose I can trust what he says. Should I now change my probability for A to 1/5?

(The puzzle can be made more puzzling and hence more entertaining by supposing that my opponent then makes another announcement: "I have the ace of spades". Should I then change my probability for his having both aces to 1/3? What difference should it make this he has now told me a suit? After all, if he had an ace, he could always name a suit. But let us forego this additional entertainment for now.)

The difficulty with changing my probability to 1/5 (or with giving any probability whatsoever without having more information) is that my opponent decides whether to tell me that B is true, and I can get into trouble (lose money on average) if he makes this decision maliciously. For example, 1/5 is certainly not the right probability for me to give if his policy is to tell me that he has at least one ace only when he does not hold both aces.

Different mathematicians and philosophers resolve the puzzle in different ways, and the diversity of their explanations has always seemed to me evidence of the depth of the problem. But rather than review a catalog of explanations and offer a list of references, as I did in Shafer (1985), I will just mention these maxims:

- You should always take account of all your information, and your new information includes not only what you learn but also how you learned it.
- You can use conditional probability to take account of new information only if this new information—all of it—corresponds to an event in your model.

These maxims tell us that we can deal with the puzzle using conditional probability only if we extend the model by giving a rule (or probabilities) for what the opponent will tell me. How my probability should change will depend on this model. For example, if the model is that the opponent tells me that he has at least one ace if and only if he has an ace and a two, then my new probability that he has both aces should be zero, not 1/5. On the other hand, if it is agreed at the outset that he will tell me whether he has at least one ace, then 1/5 is correct. These two cases are contrasted graphically in Figure 2. They are obviously not the only possible cases.

In practice, we usually do not know rules by which new information is conveyed to us, and it is often reasonable to doubt whether there are such rules. In other words, structures such as those in Figure 2 are unknown to us or even do not exist. De Finetti made it clear (especially de Finetti 1973) that he considered such structures rare and therefore felt that they could not be the basis for a theory of subjective probability. I think his position was quite reasonable in this respect. We can always invent a model for how information has come to us, but if there is little evidence for any such model, then it may make more sense to make our subjective probability judgments in some other way. As de Finetti argued, there may be no systematic way of doing it, beyond reflecting on our preferences among different risks in light of all our information and experience, conscious and unconscious.

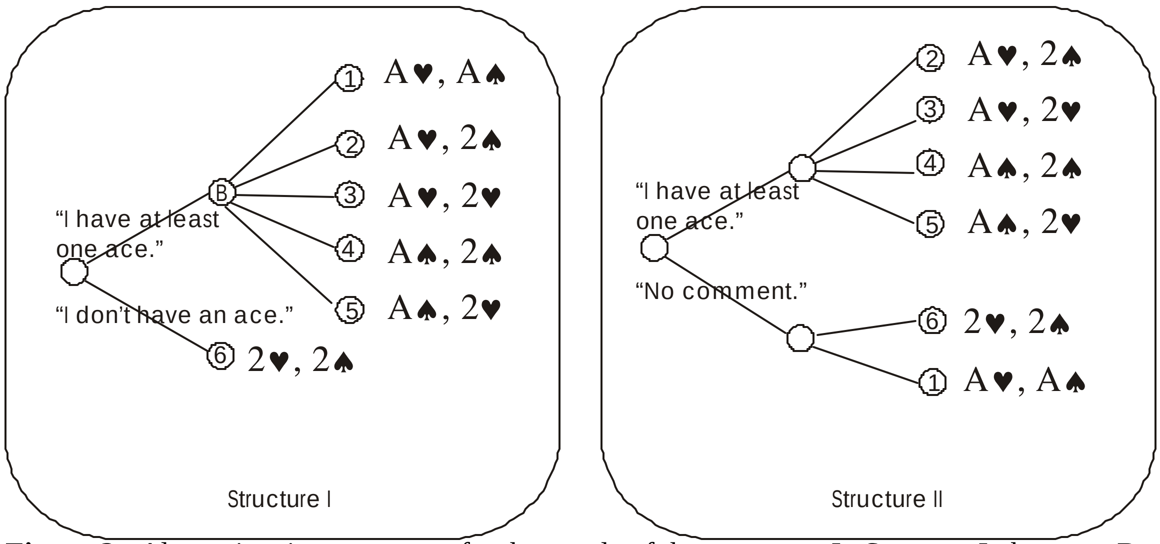

Figure 2. Alternative time structures for the puzzle of the two aces. In Structure I, the event B, identified as the subset {1,2,3,4,5} of the sample space, can also be identified as a single node in the unfolding of events or information. At this node—at this point in the story—B constitutes the totality of what has happened, and the totality of what we have learned, and so the conditional probability $P_B(A)$ becomes our new probability for whether a second event A will happen. In Structure II, the subset B = {1,2,3,4,5} does appear as a node, and hence the conditional probability $P_B(A)$ never plays this role.

One of de Finetti's arguments against reliance on structures such as those in Figure 2 for determining changes in belief (see especially de Finetti 1970) is the indefinite and effectively infinite detail in the world and our perception of it. How, in light of this infinite detail, can we pretend to identify within any formal the totality of what we have learned at a particular point in time? In the story of Figure 2, for example, we surely learn more than what is conveyed by the words "I have at least one ace." How did the opponent say these words? With what tone of voice? Did he smile? There is no end to what might influence our subjective guesses and probability judgments about the cards he holds.

In order to accommodate this indefinite potential for adding detail to our formal picture of the world, de Finetti distanced himself in an important respect from measure theory's picture of the sample space. He retained measure theory's flat and timeless notion of event, but he refused to identify an event A with a subset of a particular fixed sample space Ù, on the grounds that each of the possibilities listed in Ù can always be refined—i.e., split into more detailed possibilities. For example, possibility number 6 in our story—the possibility that the opponent holds both aces—can be split into the possibility that he holds two aces and smiles and the possibilities that he holds two aces and does not smile. Instead an event should be thought of abstractly, as an element of a

Boolean algebra of events, which can always be enlarged as we bring more topics into the conversation (de Finetti 1970).

I can only agree with de Finetti's insight about the indefinite refinability of events and also with his recognition of the impossibility of founding a theory of subjective probability on a complete model for obtaining information. And I concede that we are reduced to a flat and timeless concept of event when we give up thinking in terms of such a model. But I part company with de Finetti when he claims that his subjective theory, based on this flat and timeless concept, is preeminently and even exclusively the theory of probability. This thought was tenable in de Finetti's time because, when most students of probability were bedazzled by the new light cast by measure theory, which made a flat and timeless concept of event official. But at the distance we now stand from measure theory, from which we can see both the earlier history of probability and the possibility of new foundations (see especially the game-theoretic foundation for probability advanced by Shafer and Vovk 2001). I venture to say that dynamics is central to the intuitive picture of probability theory. The absence of dynamics from measure theory tells us that measure theory does not really provide an adequate mathematical foundation for probability. And the absence of dynamics from de Finetti's subjective theory tells us that this theory, for all it may have to offer, does not really provide an adequate philosophical foundation for probability.

5. A Notion of Event for Objective Probability and Causality

De Finetti's most fundamental and enduring insight, it seems to me, is the insight that probability, in all its ramifications and applications, is inextricably tied to the idea of betting. A probability for an event is never a property of the event alone; in order to understand the story, we must also talk about the person offering to bet. On this point he was right and the frequentists were wrong.

This does not exclude, however, an objective conception of probability, and I believe that such an objective conception underlies the use of probability in science and in our ordinary understanding of causality. The trick is reconciling objectivity with the notion of betting. Who is the offering the bet when the probability is objective?

The question almost answers itself, the difficulty is not that the answer is hidden but rather that it sounds, at first blush, rather old-fashioned and definitely out of fashion. Objective probability represents the rate at which the most informed observer we can imagine would bet. It is the probability for an ideal scientist, who knows and witnesses everything about the world that can be known and witnessed. It is the limit, as it were, of the probabilities that would be given by increasingly informed scientists.

What is the notion of event emerges from this way of understanding objective probability? It is not flat and timeless, for our ideal scientist moves through time, and his probabilities change systematically as he witnesses the unfolding of events. But we must also respect de Finetti's insight about the indefinite refinability of any picture of the world. No event, however situated it is in time, can tell us everything about what has happened so far; there is always more to say.

The challenge that I am posing here becomes clearer if we look more broadly at the intellectual superstructure that has been built around the flat and timeless notion of event that I want to replace. As I have already hinted, the sample space of measure theory and statistics is only one of the places where this notion of event lives; we also find it in the more abstract notion of a Boolean algebra, which de Finetti preferred, and in probabilistic logic. Figure 3 lays out this landscape.

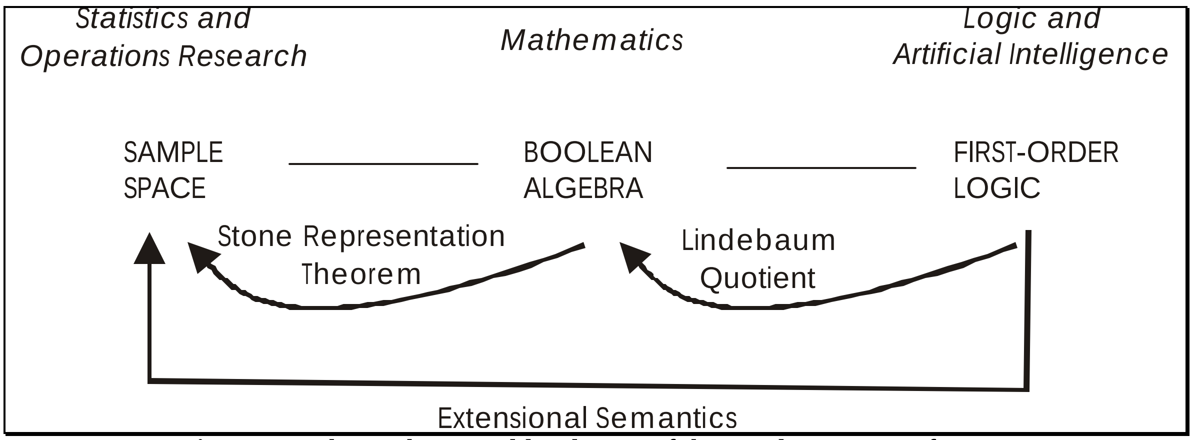

Figure 3. The mathematical landscape of the timeless notion of event.

In the theory of objective probability that I propose, sample spaces are replaced by trees, such as those that I drew in Figures 1 and 2. (Well, not always exactly like those trees; we can abstract from the limitations of pencil and paper imagine that our trees, like our sample spaces, are infinite—infinitely prolonged (time may be infinite), infinitely broad (the branching at any particular point may be continuous), and infinitely fine (time also may be continuous).)

An event in this new dynamic theory is a partial slice across the tree, formed by one or more nodes. I consider such slices, rather than individual nodes, because of the possibility of refinement, which I learned from de Finetti. A single node, when considered from a more detailed point of view, may become a slice containing several nodes, as in Figure 4.

Let me list some further aspects of the meaning and character of the concept of event in this dynamic theory.

- Events are situated in time.
- An event happens instantaneously. If we think of our ideal scientist's progress as movement along the realized path of the tree, then the event E "happens" at the instant that the ideal scientist arrives at E. (To conform with this interpretation, we should think of the event E in Figure 4 as the event that Rick begins to watch television; the watching may then continue.)

- If happens at most once as events unfold. (So we think of the event E in Figure 4 as watch beginning to watch television for the first time; this does not rule out his stopping and starting again.)
- On the other hand, the time at which E may happen is not necessarily fixed in advance. In Figure 4, for example, if E_1 can happen only at 4:00, E_2 can happen only at 5:00, and E_3 can happen only at 6:00, then E can happen at any of these times.

From a philosophical point of view, the situatedness in time is most important here; the rest is, in a sense, a product of mathematical esthetic—a way of making a clear and simple theory. The situatedness in time means, of course, that the event has more objective meaning that de Finetti gave to his events; here not every objective meaning disappears in the event's happening or not happening—turning out to be true or false.

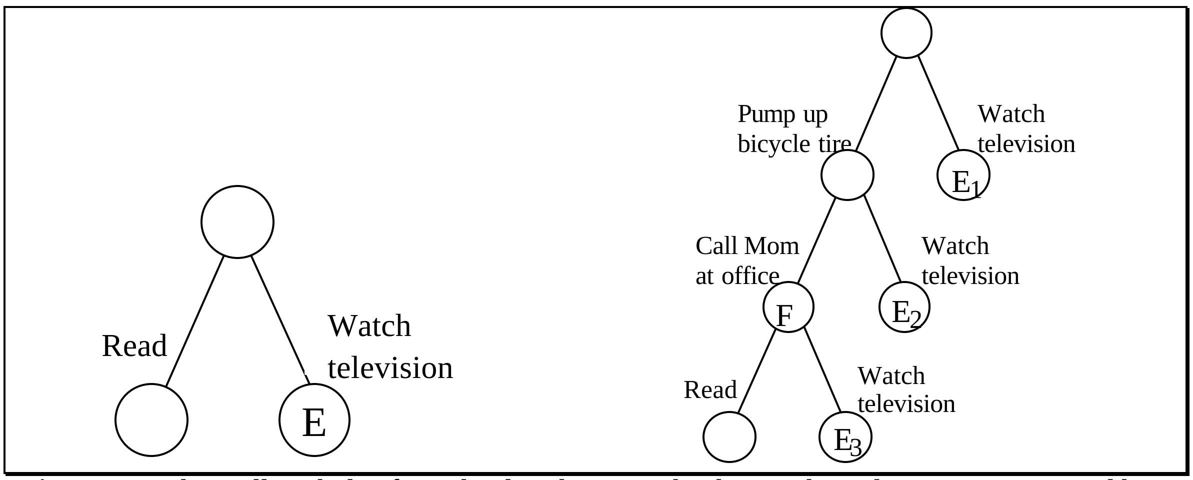

Figure 4. What will Rick do after school? The event that he watches television, represented by a single node E in the tree on the left, becomes a partial slice across the tree when it is refined, on the right. From an abstract point of view, the events E and $\{E_1,E_2,E_3\}$ are the same event.

The situated of an event E in this theory allows it to be both a bearer and giver of probability. Regarded from outside, say from a point earlier in the unfolding of events, E may or may not happen, and our ideal scientist may be able to give a probability for its happening. From the inside, at the point where the event instantaneously happens, it provides a probability for other events, and from this perspective we may wish to call it a *situation* rather than an *event*.

If two events are causally connected in such a way that one can happen only if the other happens (perhaps later, perhaps earlier or perhaps at the same time), then the two events are equivalent from de Finetti's point of view, and this means that they always have the same probability. But they may behave quite differently as situations—as identifiers of probability for other events. This is illustrated in Figure 5. This figure also begins to show how probability in the new theory differs from the additive probabilities of de Finetti's theory. In some situations, an event may fail to have a probability, perhaps because we need to refine the situation. We may need to bring in more detail about what has happened so far in order to identify an exact probability. And even this may fail: the our ideal scientist, even in the limit, may be unwilling to take gambles that define an exact probability, and so in general an event has only an upper and lower objective probability in a situation. But this is a topic for another day.

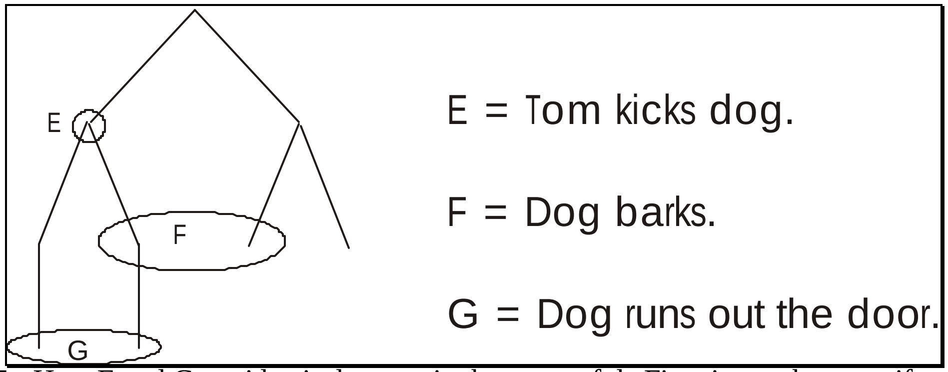

Figure 5. Here E and G are identical events in the sense of de Finetti: one happens if and only if the other happens. As this implies, they are equivalent as bearers of probability. If the probability of E is 75% at the beginning of the story (at the top of the treee), then the probability of G is also 75% at that point. But E and G are not necessarily identical as givers of probability. In situation E, F is possible but not necessary, and hence we may suppose that it has a probability strictly between 0 and 1. But in situation G, F has already happened or failed, and so its probability is either 0 or 1, depending on which more refined situation we are actually in.

The program of research to which I am introducing you here involves not only a reinterpretation of probability in trees but also a generalization of the entire structure in Figure 3, as indicated in Figure 6. We abstract of the idea of a situated instantaneous event from the context of a tree in the same way that we abstract the idea of a timeless event from the context of a sample space: by axiomatizing it. In the case of the timeless event, this produces the axioms for Boolean algebra. In the case of the situated event, this produces axioms for a generalization of Boolean algebra, which I call an event space. The best axiomatization so far for event spaces is given in my paper with Gillett and Scherl, which you can download from the internet. If time allows, I will conclude my talk by explaining some of these axioms.

Figure 6. The mathematical landscape of the situated notion of event. Not indicated are relations among the components, which generalize those of Figure 3. For more details about the implied research program, see Shafer, Gillett, and Scherl (2000).

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