Propagating Belief Functions in AND-Trees†

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ABSTRACT

We describe a simple method for propagating belief functions in AND-trees. We exploit the properties of AND-trees to make our method simpler than the general method discussed by Shenoy and Shafer, and Dempster and Kong. We illustrate our method for aggregation of evidence in a financial audit.

Key Words: Aggregation of evidence, propagation of belief functions in AND-trees

I. INTRODUCTION

The main goal of this article is to describe a simple method for propagating belief functions in AND-trees. The propagation of belief functions in general networks has been discussed by several researchers (see, e.g., Shenoy and Shafer [1], Shafer, Shenoy and Mellouli [2], Dempster and Kong [3], Shenoy and Shafer [4], and Shenoy [5]). There are several computer implementations of these methods (see, e.g., Zarley, Hsia and Shafer [6], Hsia and Shenoy [7], Almond [8], Saffiotti and Umkehrer [9], and Xu [10]). The general propagation method is quite complex for numerical computations without the aid of a computer system. As described in this article, the general method can be simplified when the network is an AND-tree. The simplification is achieved by exploiting certain properties of AND-trees.

AND-trees occur frequently in financial auditing. In this domain, we have a main objective which is met if and only if several sub-objectives are met. Figure 1 shows a simple evidential AND-tree in a financial audit. A box with rounded corners represents a variable node, and a rectangular box represents an evidence node. Edges connecting an evidence node to variable nodes indicates the domain of the evidence. A circle with the symbol & represents an AND node.

An AND-tree is a rooted tree consisting of variable nodes and AND nodes. Variables in AND-trees are assumed to be binary, i.e., a variable that has only two possible values. For example, an account on the balance sheet is either fairly stated or materially misstated, and an audit objective of an account is either met or not met. Such binary variables are common in auditing.

Each AND node has exactly one edge leading to it. An AND node implies that the variable on the left (toward the root) is true if and only if the variables on the right (away from the root) are true. Furthermore, we assume each item of evidence bears on only one variable node in the AND-tree.

Figure 1. An Evidential AND-Tree for Accounts Receivable with only Two Audit Objectives (Procedures 1-7 are described in Table 1).

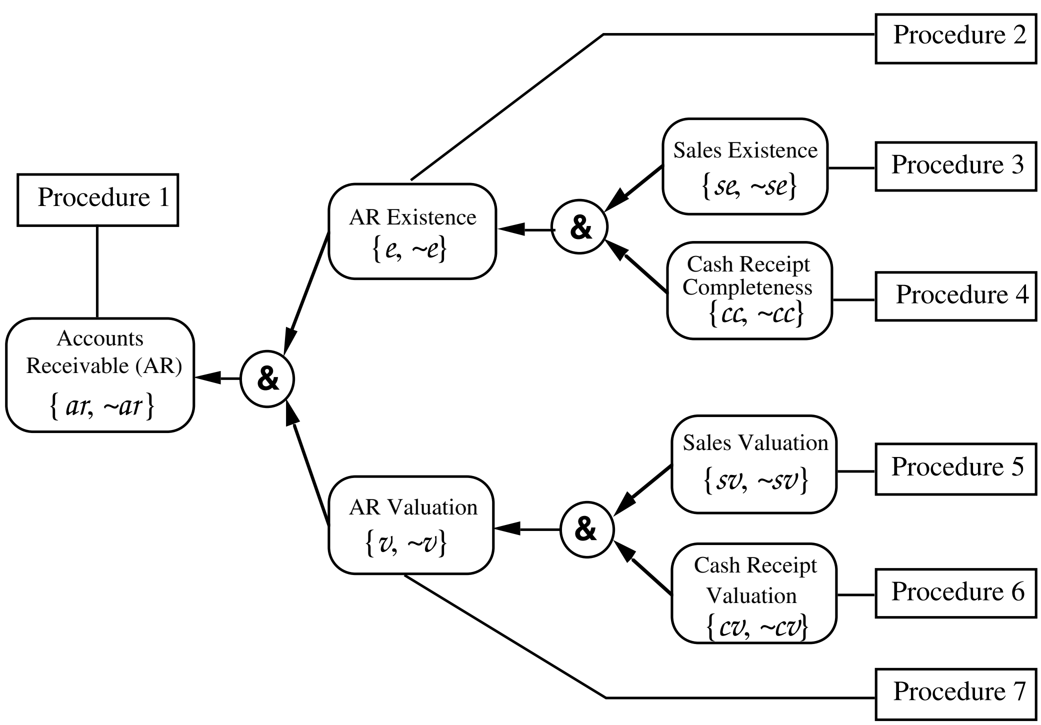
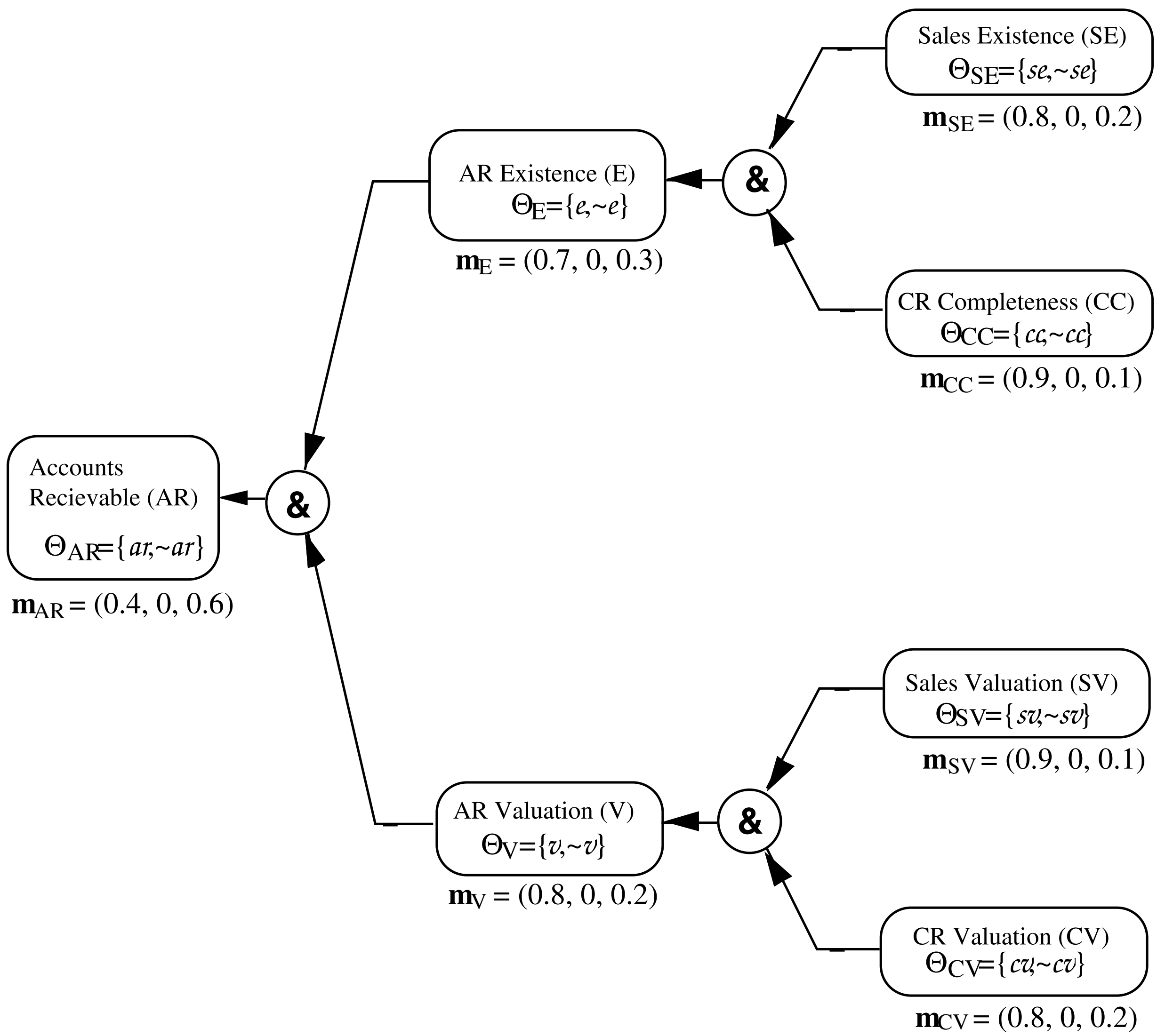


Table 1. Procedures used in Figure 1 (see Arens and Loebbecke [11] for details)

Procedure 1 -	Review accounts receivable trial balance for large and unusual receivables. Also, calculate ratios indicated in carry-forward working papers and follow up on any significant changes from prior years.
Procedure 2 -	Confirm accounts receivable using positive confirmations. Confirm all amounts over $5,000 and a nonstatistical sample of the remainder to see whether these accounts exist.
Procedure 3 -	(i) Trace sales journal entries to duplicate sales invoices and shipping documents. (ii) Trace shipping documents to entry of shipments in perpetual inventory records. (iii) Trace sales journal entries to sales orders for credit approval and shipping authorization.
Procedure 4 -	Trace from remittance or prelisting to cash receipt journal.
Procedure 5 -	(i) Recompute information on sales invoices. (ii) Trace details on sales invoices to price lists, and customers' orders.
Procedure 6 -	Perform proof of cash receipts.
Procedure 7 -	Discuss with the credit manager the likelihood of collecting older accounts. Examine subsequent cash receipts and the credit file on all accounts over 120 days and evaluate whether the receivables are collectible. Also, evaluate whether the allowance is adequate after performing other audit procedures relating to collectibility of receivables.

In general, uncertainties involved with audit evidence can be expressed in terms of belief functions (see, e.g., Shafer, Shenoy, and Srivastava [12], Shafer and Srivastava [13], Srivastava and Shafer [14]). Aggregation of audit evidence is, in fact, a problem of propagating belief functions in a network (see, e.g., Srivastava [15]). For propagating belief functions, we convert the evidential network into a network of only the variables with the respective belief functions on each variable arising from the corresponding item of evidence. Figure 2 shows such a network for Figure 1.

Figure 2. The AND-Tree in Figure 1 with **m**-values at Each Node. (These **m**-values represent only the evidence bearing directly on each node.)

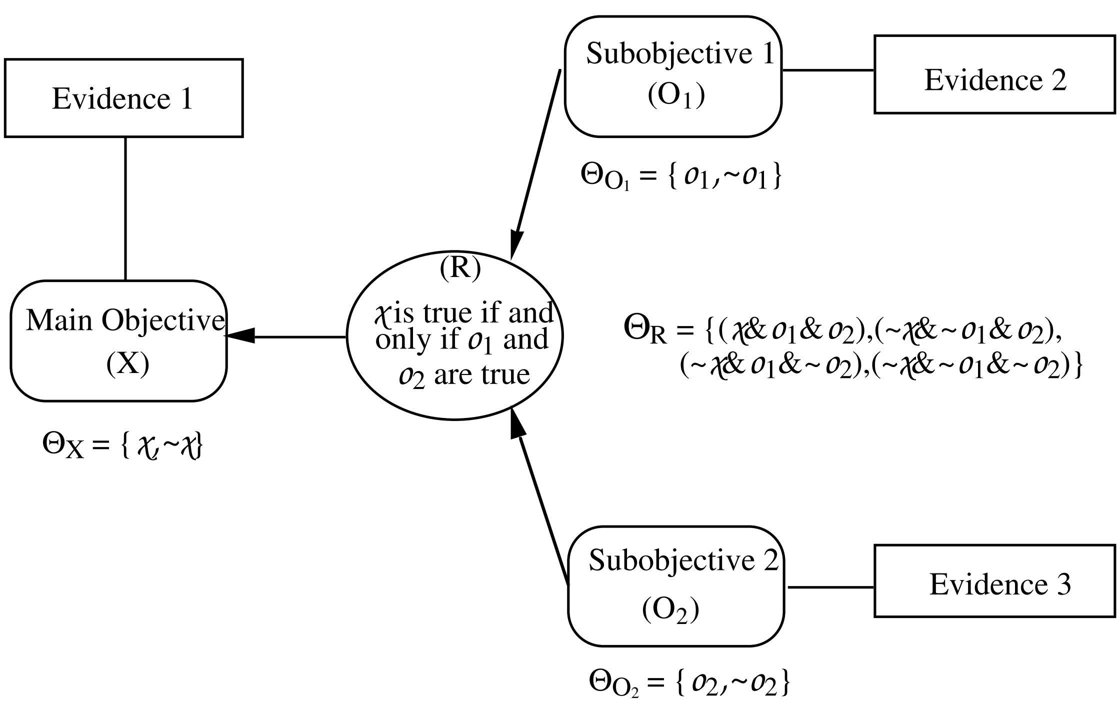


The remainder of this paper is divided into three sections. In section II, we provide the general formula for propagating belief functions in AND-trees and discuss the results. In section III, we use the general results of Section II to show how they can be used to aggregate evidence in a financial audit. In Section IV, we summarize the results and provide a conclusion. Finally, in Section V, we provide proofs of the two propositions described in Section II.

II. PROPAGATION OF BELIEF FUNCTIONS IN AND-TREES

In order to derive the general results, consider a simple evidential network of three variables X, O_1 and O_2, as shown in Figure 3. Assume that these variables are binary. We will use upper-case letters to represent names of variables and lower-case letters in script to represent their values. For example, X is the name of a variable and x, and $\sim x$ are its two values; x means that X is true and $\sim x$ means that X is not true. Thus, the corresponding frames are: $\Theta_X = \{x, \sim x\}$, $\Theta_{O1} = \{o_1, \sim o_1\}$, and $\Theta_{O2} = \{o_2, \sim o_2\}$. As shown in Figure 3, we assume that the two variables, O_1 and O_2, are related to the variable X through an AND node, i.e., $X = x$ if and only if $O_1 = o_1$ and $O_2 = o_2$. This relationship is incorporated in our analysis by assuming that the frame of the relational node R is $\Theta_R = \{(x, o_1, o_2), (\sim x, \sim o_1, o_2), (\sim x, o_1, \sim o_2), (\sim x, \sim o_1, \sim o_2)\}$.[1]

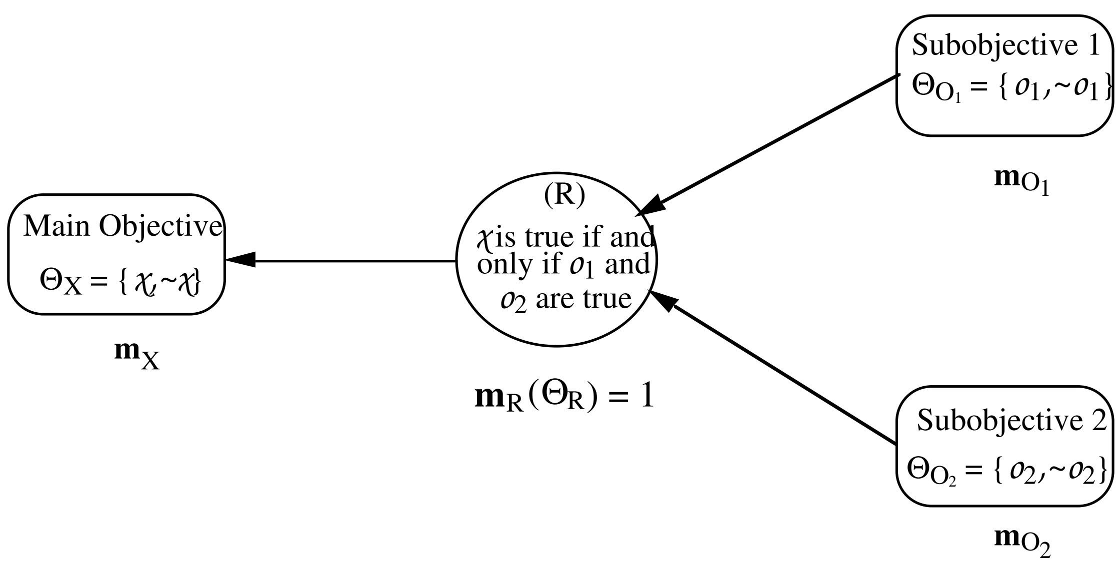
Figure 3. An Evidential AND-Tree with Three Variable Nodes



Evidence for a variable is represented by a basic probability assignment (bpa) function. For a binary variable X with frame $\{\chi, \sim\chi\}$, a bpa function **m** is completely defined by three non-negative numbers that add to 1, $\mathbf{m}(\{\chi\})$, $\mathbf{m}(\{\sim\chi\})$, and $\mathbf{m}(\{\chi, \sim\chi\})$. We refer to these numbers as **m**-values. To simplify notation, we write $\mathbf{m}(\chi)$ instead of $\mathbf{m}(\{\chi\})$, and $\mathbf{m}(\sim\chi)$ instead of $\mathbf{m}(\{\sim\chi\})$. In the figures, a bpa function **m** for X is shown as a 3-vector $(\mathbf{m}(\chi), \mathbf{m}(\sim\chi), \mathbf{m}(\{\chi, \sim\chi\}))$.

In our examples, we have assumed one item of evidence for each variable. Thus, we have one set of **m**-values for each variable (see Figures 2, 5, 6, and 7). For propagation purposes, once we determine the **m**-values representing evidence at different nodes, we represent the network without the evidence nodes as shown in Figure 4. In general, we can assume more than one item of evidence for each node. In such cases, we needs to first combine the items of evidence at each node before doing the propagation.

Figure 4. An AND-Tree with Three Nodes.



Notice that Evidence 1 bearing directly on node X will impact indirectly both nodes, O_1 and O_2. Similarly, Evidence 2 and Evidence 3 together will impact node X but neither Evidence 2 nor Evidence 3 by themselves will affect X because of the AND relationship. Also, since O_1 and O_2 are connected to X through AND, evidence at O_1 alone will not affect O_2 and evidence at O_2 alone will not affect O_1. These properties are the special features of AND-trees. In general trees, each node is indirectly affected by the evidence at the other nodes.

In order to describe the propagation process, we need some notation. Suppose X is a variable in an AND-tree. Then, $\mathbf{m}_X$ denotes the bpa function representation of evidence that bears on X.

$\mathbf{m}^t_X$ is a bpa function for X representing the marginal of the combination of all evidence in the AND-tree, i.e., $\mathbf{m}^t_X = (\oplus\{m_Y \mid Y \text{ is a variable in the AND-tree}\})^{\downarrow X}$. Our goal is to compute $\mathbf{m}^t_X$ for all nodes X given $\mathbf{m}_Y$ for all nodes Y in the AND-tree.

Finally, $\mathbf{m}_{X\leftarrow\{O_1, \ldots O_n\}}$ denotes the bpa function for X representing the marginal of the combination of bpa functions $\mathbf{m}_{O_i}$ for i = 1, ..., n. To keep our notation short, we will abbreviate $\mathbf{m}_{X\leftarrow\{O_1, \ldots O_n\}}$ to $\mathbf{m}_{X\leftarrow \text{all O's}}$. And in an AND-tree with variables X, O_1, ..., O_n, we will abbreviate $\mathbf{m}_{O_i\leftarrow\{X, O_1, \ldots O_{i-1}, O_{i+1}, \ldots, O_n\}}$ by $\mathbf{m}_{O_i\leftarrow X\&\text{all other O's}}$.

Proposition 1 (*Propagation of **m**-values from sub-objectives to the main objective*): The resultant **m**-values propagated from n sub-objectives (O_i, i = 1, 2, . . . n) to the main objective X in an AND-tree are given as follows.

$$\mathbf{m}_{X\leftarrow \text{ all O's}}(x) = \prod_{i=1}^{n} \mathbf{m}_{O_i}(o_i) \tag{1}$$

$$\mathbf{m}_{X\leftarrow \text{all O's}}(\sim x) = 1 - \prod_{i=1}^{n} [1 - \mathbf{m}_{O_i}(\sim o_i)] \tag{2}$$

and

$$\mathbf{m}_{X\leftarrow \text{all O's}}(\{x, \sim x\}) = 1 - \mathbf{m}_{X\leftarrow \text{all O's}}(x) - \mathbf{m}_{X\leftarrow \text{all O's}}(\sim x) \tag{3}$$

The above results conform to our intuition. Equation 1 states that the degree of belief that the main objective is met is the product of the beliefs of the sub-objectives. This is a consequence of the AND relation between the main objective and the sub-objectives. This formula corresponds to the product rule in probability theory. Equation 2 can be explained as follows. Notice that $1 - \mathbf{m}_{O_i}(\sim o_i)$ is the degree of plausibility that sub-objective i is met. Equation (2) states that the degree of belief that the main objective is not met is 1 minus the product of the plausibilities that the sub-objectives is met. This is again a consequence of the AND relation. The main objective is not met if and only if sub-objective 1 is not met OR sub-objective 2 is not met, etc. Again, this rule corresponds to the product rule in probability theory.

Proposition 2 (*Propagation of **m**-values to a given sub-objective from the main objective and the other sub-objectives*): The resultant **m**-values propagated to a given sub-objective O_i from the main objective X and the other n-1 sub-objectives in an AND-tree are given as follows.

$$\mathbf{m}_{O_i \leftarrow X\&\text{All other O's}}(o_i) = K_i^{-1}\mathbf{m}_X(x)\prod_{j \neq i}[1 - \mathbf{m}_{O_j}(\sim o_j)] \ , \tag{4}$$

$$\mathbf{m}_{O_i \leftarrow X\&\text{All other O's}}(\sim o_i) = K_i^{-1}\mathbf{m}_X(\sim x)\prod_{j \neq i}\mathbf{m}_{O_j}(o_j) \ , \tag{5}$$

$$\mathbf{m}_{O_i \leftarrow X\&\text{All other O's}}(\{o_i, \sim o_i\}) = 1 - \mathbf{m}_{O_i \leftarrow X\&\text{All other O's}}(o_i) - \mathbf{m}_{O_i \leftarrow X\&\text{All other O's}}(\sim o_i). \tag{6}$$

where K_i is the renormalization constant which is given by $K_i = [1 - \mathbf{m}_X(x)C_i]$, where C_i is given by

$$C_i = 1 - \prod_{j \neq i}[1 - \mathbf{m}_{O_j}(\sim o_j)] \ . \tag{7}$$

Again, the above results are intuitive. Equation (4) suggests that sub-objective O_i is met when the main objective is met and the other sub-objectives are either met or we do not know whether they are met. Equation (5) means that sub-objective O_i is not met when the main objective is not met and all other sub-objectives are met. The conflict term, $\mathbf{m}_X(x)C_i$, in K_i is also intuitive. It suggests that a conflict exists when the main objective is met and at least one of the other sub-objectives (other than O_i) is not met (see Equation (7)).

Discussion of the General Results

We have discussed two types of propagation of **m**-values in a AND-trees. The first one is from the sub-objectives to the main objective. The second one is to a given sub-objective from the main objective and the other sub-objectives. These results conform to our intuition. For example, in the first case, when individual sub-objectives are each true with a certain degree of assurance, then the product of all these values should give the assurance for the main objective to be true (Equation 1). As mentioned earlier, this is similar to the product rule in probability theory.

The second case is interesting. It suggests that the effects on a given sub-objective of **m**-values at the main objective will depend on the type and strength of evidence (**m**-values) for the rest of the sub-objectives. For example, if all other sub-objectives have positive support, i.e., $\mathbf{m}_{O_j}(o_j) > 0$ for all $j \neq i$, then any evidence that the main objective is not met (i.e., $\mathbf{m}_X(\sim x) > 0$) will provide positive support that the sub-objective O_i is not met (5). However, when we have no evidence that other sub-objectives have been met ($\mathbf{m}_{O_j}(o_j) = 0$), the evidence that the main objective is not met ($\mathbf{m}_X(\sim x) > 0$) would provide no support for the objective O_i not being met, i.e., $\mathbf{m}_{O_i \leftarrow X\&\text{All other O's}}(\sim o_i) = 0$. Both the cases will be used in Section III for combining audit evidence.

III. AGGREGATION OF AUDIT EVIDENCE IN AND-TREES

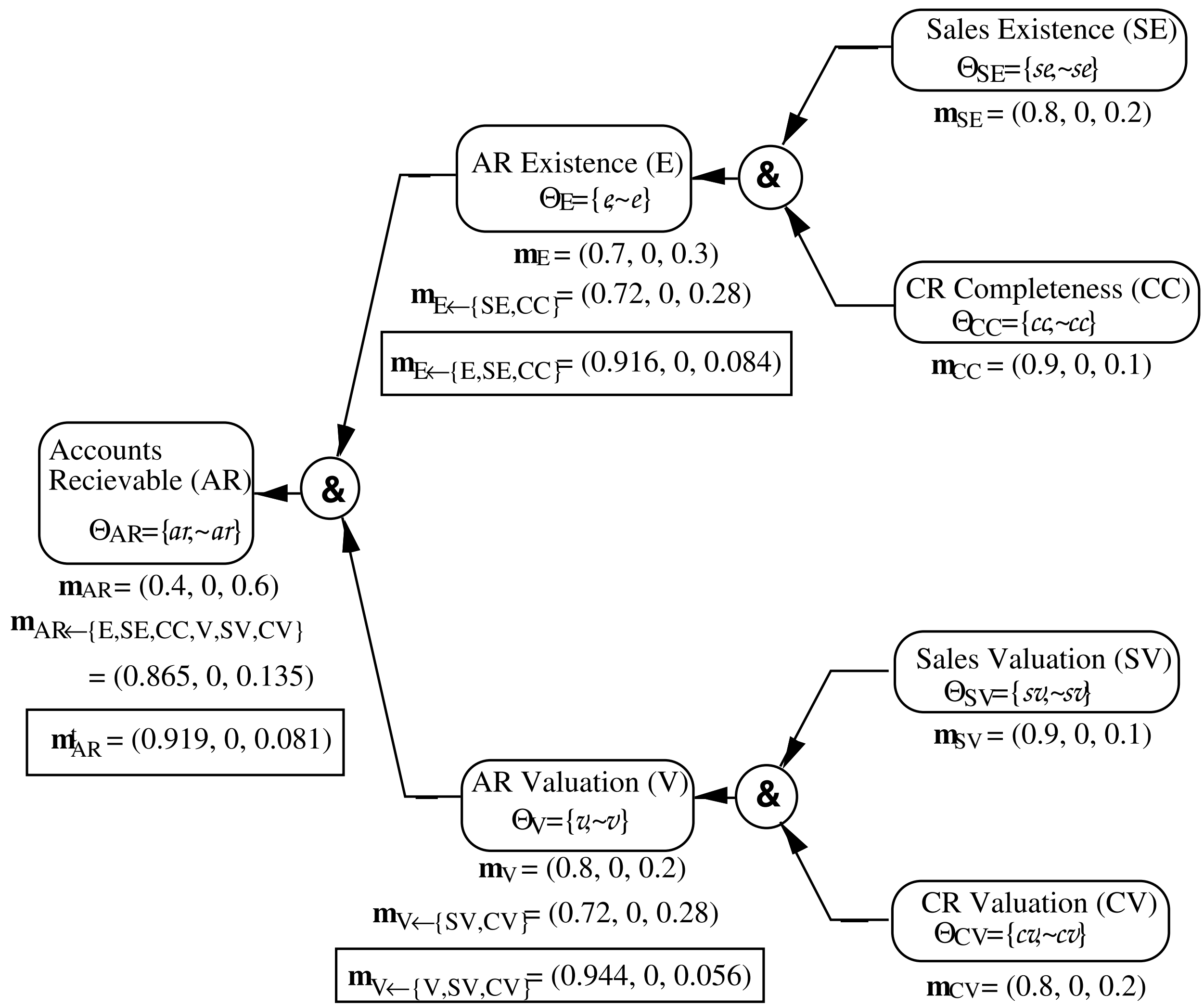
In this section, we illustrate how the two propositions discussed in Section II can be used to combine evidence in an audit for planning and evaluation. For simplicity of computations we will use the structure of evidence presented in Figure 1. We assume that the accounts receivable account in Figure 1 has only two audit objectives: Existence (E) and Valuation (V), and only one item of evidence for each node.[2] Thus, we have seven nodes and seven items of evidence. In fact, these items of evidence can be considered to be the procedures performed by the auditor (see Table 1 for details). Let us assume that the auditor has made judgments about the level of support obtained from these procedures for the respective nodes. We represent these values just below the respective nodes in Figure 2.

We want to determine the overall support for each node in Figure 2 as a result of aggregating all the evidence. For this purpose, we need to propagate **m**-values defined at each node through the entire tree and combine the **m**-values received by each node from its neighbors with the **m**-values defined at the node. The following sub-sections provide the results of aggregation at three different levels.

Level of Support for Accounts Receivable

In order to determine the overall assurance that the accounts receivable balance is fairly stated, i.e., *ar* is true (Figure 2), we must aggregate all the evidence gathered, evidence at the account level, at the audit objectives level, and at the sales and cash receipts levels (Procedures 1 - 7 in Table 1). This is achieved by propagating **m**-values from the sub-objectives to the main objective 'AR' in steps. First, we propagate **m**-values from SE and CC to E , and from SV and CV to V using Proposition 1. This yields $\mathbf{m}_{E\leftarrow\{SE,CC\}}$ and $\mathbf{m}_{V\leftarrow\{SV,CV\}}$ as listed under the respective nodes in Figure 5.

Figure 5. Overall Support for Accounts Receivable.



The second step is to combine $\mathbf{m}_{E\leftarrow\{SE, CC\}}$ with $\mathbf{m}_E$, and $\mathbf{m}_{V\leftarrow\{SV, CV\}}$ with $\mathbf{m}_V$. We use Dempster's rule to combine the two sets of bpa functions. The resulting **m**-values, $\mathbf{m}_{E\leftarrow\{E,SE,CC\}}$ and $\mathbf{m}_{V\leftarrow\{V,SV,CV\}}$, are given in rectangular boxes under the respective nodes. The third step is to take these bpa functions and propagate them to AR. We use again Proposition 1 to determine the **m**-values received by AR. These **m**-values are represented by $\mathbf{m}_{AR\leftarrow\{E,V,SE,CC,SV,CV\}}$ below AR node in Figure 5. Finally, we use again Dempster's rule to combine

$\mathbf{m}_{AR\leftarrow\{E,V,SE,CC,SV,CV\}}$ with $\mathbf{m}_{AR}$. The resulting (total) bpa function is as given in the rectangular box under AR (Figure 5):

$$\mathbf{m}^t_{AR} = (0.919, 0, 0.081).$$

By definition, the corresponding belief function is:

$$\mathbf{Bel}^t_{AR}[ar] = 0.919,\ \mathbf{Bel}^t_{AR}[\sim ar] = 0,\ \text{and}\ \mathbf{Bel}^t_{AR}[\{ar, \sim ar\}] = 1.$$

This result suggests that when all the evidence with their respective strengths is aggregated, the overall assurance that the accounts receivable balance will be fairly stated, i.e., *ar* will be true, would be 0.919. Given the inputs, there is no support for ~*ar*. Support for ~*ar* would result, however, if there were evidence against any of the objectives.

Support at the Audit Objective Level

In this subsection, we determine the overall level of support for each audit objective, E and V. Again the evidence collected at the account level and the transaction level will be relevant.

Figure 6. Overall Support at the Audit Objective Level (Nodes E and V).

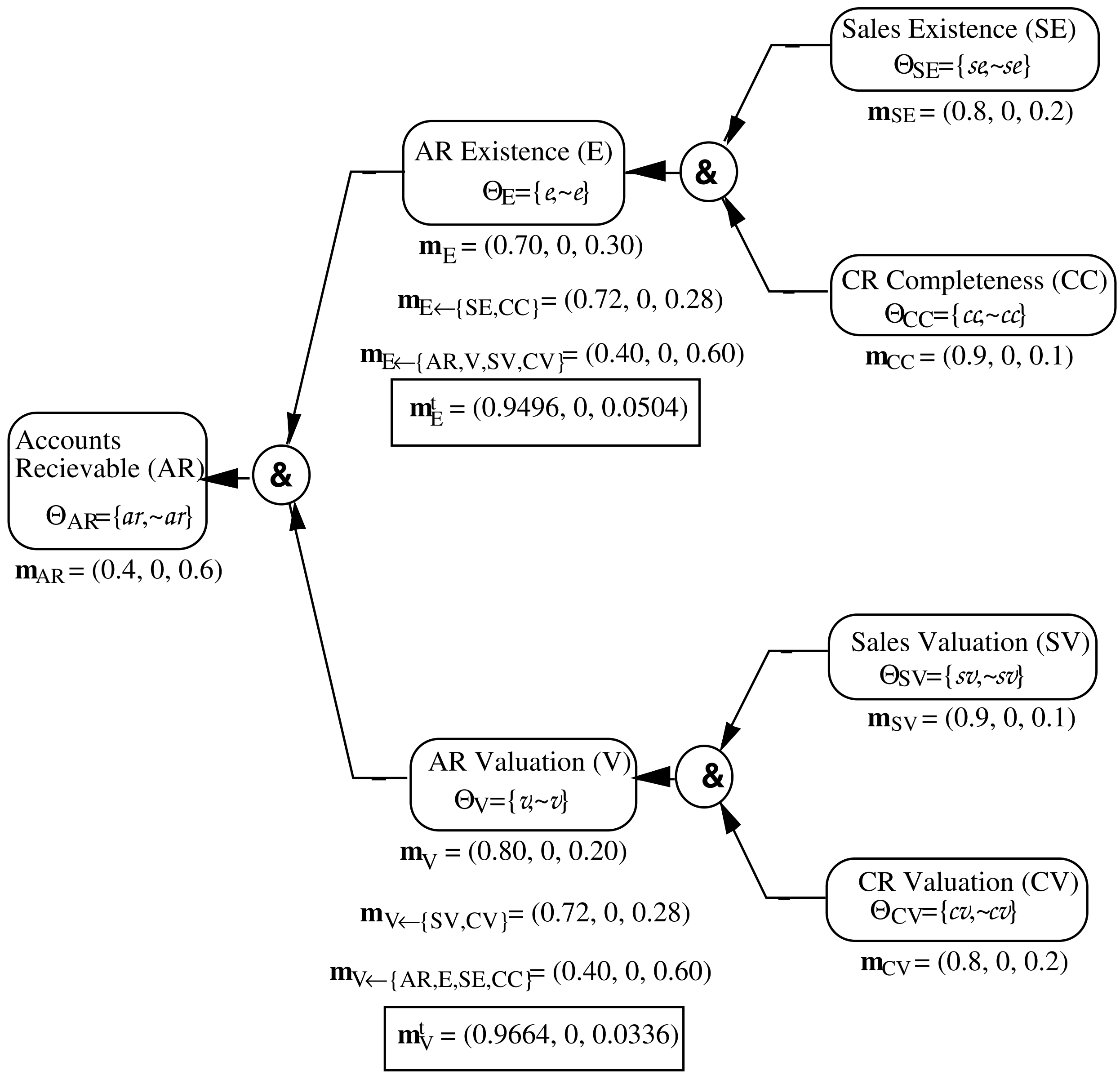


Figure 6 represents the propagation results for this case. As one can see, **m**-values from AR and V will be propagated to E and from AR and E to V using Proposition 2. Also, **m**-values from SE and CC will be propagated to E, and from SV and CV to V using Proposition 1. We have listed the resulting **m**-values ($\mathbf{m}_{E\leftarrow\{AR,V,SV,CV\}}$, $\mathbf{m}_{V\leftarrow\{AR,E,SE,CC\}}$, $\mathbf{m}_{E\leftarrow\{SE,CC\}}$, and

$\mathbf{m}_{V\leftarrow\{SV,CV\}}$) under the respective nodes in Figure 6. Now, we combine the three **m**-values at each audit objective and obtain the following result:

AR Existence node: $\mathbf{m}^{t}_{E}$ = (0.9496, 0, 0.0504).

AR Valuation node: $\mathbf{m}^{t}_{V}$ = (0.9664, 0, 0.0336).

The corresponding belief functions are given by:

$$\mathbf{Bel}^{t}_{E}[e] = 0.9496,\ \mathbf{Bel}^{t}_{E}[\sim e] = 0, \text{ and } \mathbf{Bel}^{t}_{E}[\{e, \sim e\}] = 1.$$

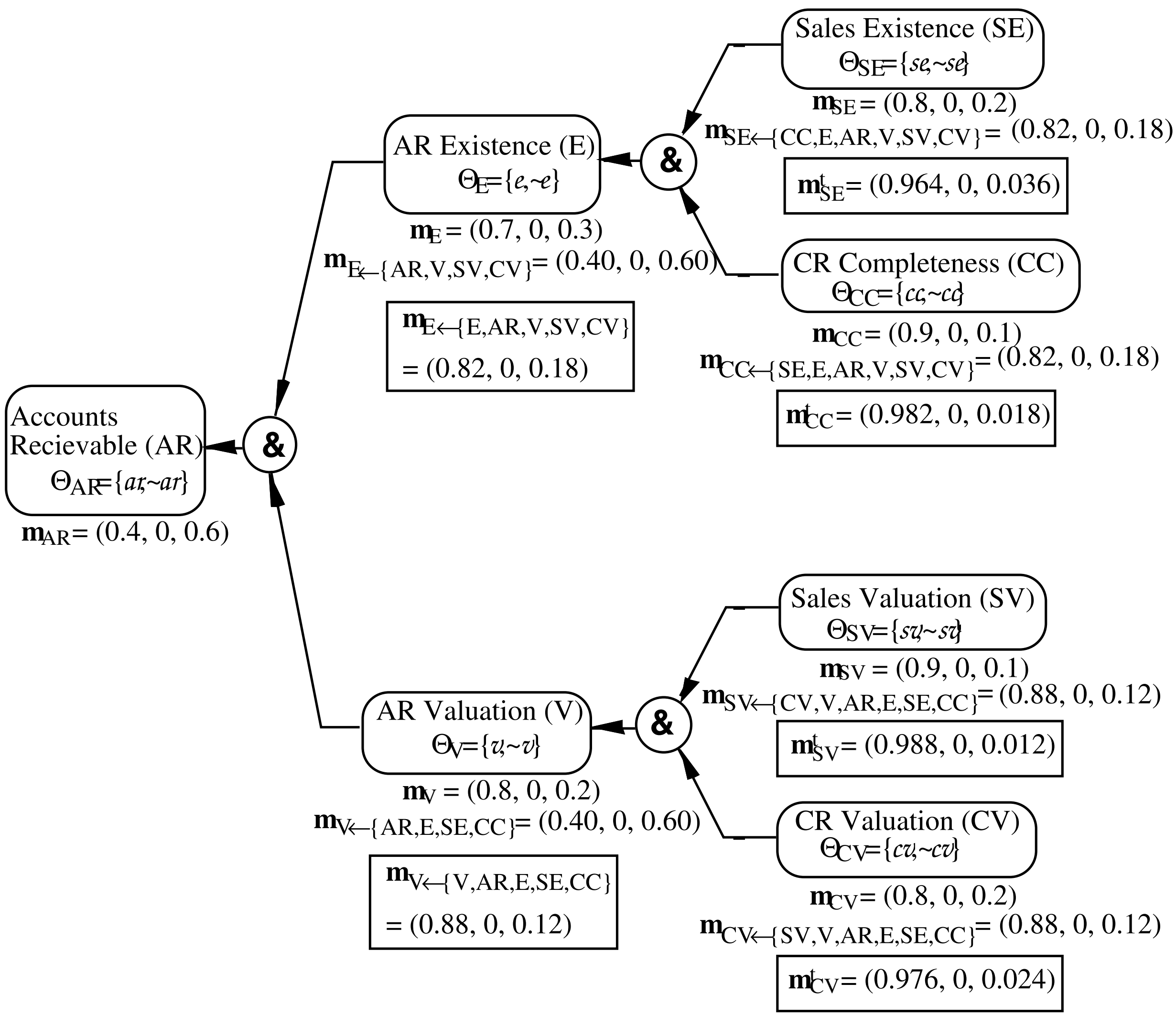
and

$$\mathbf{Bel}^{t}_{V}[v] = 0.9664,\ \mathbf{Bel}^{t}_{V}[\sim v] = 0, \text{ and } \mathbf{Bel}^{t}_{V}[\{v, \sim v\}] = 1.$$

Support at the Transaction Level

In this subsection, we discuss the aggregation of evidence at the transaction level in Figure 7. As evident from Figure 7, as a first step, we propagate **m**-values from AR and V to E and from AR and E to V. We used Proposition 2 earlier for this part and obtained $\mathbf{m}_{E\leftarrow\{AR,V,SV,CV\}}$ and $\mathbf{m}_{V\leftarrow\{AR,E,SE,CC\}}$.

Figure 7. Overall Support at the Sales and Cash Receipts levels.



As the second step, we combine the two sets of **m**-values, $\mathbf{m}_{E}$ and $\mathbf{m}_{E\leftarrow\{AR,V,SV,CV\}}$ at E, and, $\mathbf{m}_{V}$ and $\mathbf{m}_{V\leftarrow\{AR,E,SE,CC\}}$ at V. This step yields $\mathbf{m}_{E\leftarrow\{E,AR,V,SV,CV\}}$ and $\mathbf{m}_{V\leftarrow\{V,AR,E,SE,CC\}}$, respectively, as given in rectangular boxes below the corresponding objectives. In the third step, to obtain the total **m**-values at SE, we combine $\mathbf{m}_{E\leftarrow\{E,AR,V,SV,CV\}}$ at E with $\mathbf{m}_{CC}$ at CC and propagate to SE using Proposition 2. This yields $\mathbf{m}_{SE\leftarrow\{CC,E,AR,V,SV,CV\}}$. Finally, we combine $\mathbf{m}_{SE}$ with $\mathbf{m}_{SE\leftarrow\{E,AR,V,SV,CV,CC\}}$ to obtain $\mathbf{m}^{t}_{SE}$, at SE. Similar procedures are performed in step three for other nodes at the transaction level. The resulting $\mathbf{m}^t$-values for each transaction node are given in a box below the node.

IV. CONCLUSION

We have stated two propositions for propagating belief functions in AND-trees and have illustrated the use of these propositions in aggregating various items of evidence in a financial audit. The method discussed in the article can be easily programmed using a spreadsheet to automate computations. This method is simpler than the general method described by Shenoy and Shafer [1] and Dempster and Kong [3].

V. PROOFS

Proof of Proposition 1

We will use the network in Figure 4 to demonstrate how the general results can be obtained. As mentioned earlier, the two sub-objectives O_1 and O_2 in Figure 4 are connected to the main objective X through an AND node.

The **m**-values defined at node O_1 are:

$$\mathbf{m}_{O_1}(o_1),\ \mathbf{m}_{O_1}(\sim o_1),\ \text{and}\ \mathbf{m}_{O_1}(\{o_1, \sim o_1\}).$$

The vacuous extension of these **m**-values onto the frame of node R yields the **m**-values being sent to node R from node O_1:[3]

$$\mathbf{m}_{R\leftarrow O_1}(\{(x,o_1,o_2), (\sim x,o_1,\sim o_2)\}) = \mathbf{m}_{O_1}(o_1),$$
$$\mathbf{m}_{R\leftarrow O_1}(\{(\sim x,\sim o_1,o_2), (\sim x,\sim o_1,\sim o_2)\}) = \mathbf{m}_{O_1}(\sim o_1),$$
$$\mathbf{m}_{R\leftarrow O_1}(\Theta_R) = \mathbf{m}_{O_1}(\Theta_{O_1}), \qquad (8)$$

and **m**-values for all other subsets of Θ_R are zero.

Similarly, **m**-values at node O_2 is vacuously extended onto the frame of node R yielding the **m**-values being sent to node R from node O_2:

$$\mathbf{m}_{R\leftarrow O_2}(\{(x,o_1,o_2), (\sim x,\sim o_1,o_2)\}) = \mathbf{m}_{O_2}(o_2),$$

$$\mathbf{m}_{R\leftarrow O_2}(\{(\sim x,o_1,\sim o_2), (\sim x,\sim o_1,\sim o_2)\}) = \mathbf{m}_{O_2}(\sim o_2),$$

$$\mathbf{m}_{R\leftarrow O_2}(\Theta_R) = \mathbf{m}_{O_2}(\Theta_{O_2}), \tag{9}$$

and, again, **m**-values for all other subsets of Θ_R are zero.

Now, the three sets of **m**-values at R, one defined at R ($\mathbf{m}_R(\Theta_R) = 1$) and the other two received from O_1 and O_2 (Equations 8 and 9) are combined using Dempster's rule [16]. The resulting **m**-values are marginalized onto the frame of node X. This represents the combined **m**-values obtained by X from O_1 and O_2. Since $\mathbf{m}_R(\Theta_R) = 1$, the combination of $\mathbf{m}_R$, $\mathbf{m}_{R\leftarrow O_1}$, and $\mathbf{m}_{R\leftarrow O_2}$ will be the same as the combination of $\mathbf{m}_{R\leftarrow O_1}$ and $\mathbf{m}_{R\leftarrow O_2}$. Thus, using Dempster's rule to combine $\mathbf{m}_{R\leftarrow O_1}$, and $\mathbf{m}_{R\leftarrow O_2}$, one obtains the resulting **m**-values at R as given below:

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(x,o_1,o_2)$$
$$= \mathbf{m}_{R\leftarrow O_1}(\{(x,o_1,o_2), (\sim x,o_1,\sim o_2)\})\, \mathbf{m}_{R\leftarrow O_2}(\{(x,o_1,o_2), (\sim x,\sim o_1,o_2)\})$$
$$= \mathbf{m}_{O_1}(o_1)\mathbf{m}_{O_2}(o_2),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\sim x,o_1,\sim o_2)$$
$$= \mathbf{m}_{R\leftarrow O_1}(\{(x,o_1,o_2), (\sim x,o_1,\sim o_2)\})\, \mathbf{m}_{R\leftarrow O_2}(\{(\sim x,o_1,\sim o_2), (\sim x,\sim o_1,\sim o_2)\})$$
$$= \mathbf{m}_{O_1}(o_1)\mathbf{m}_{O_2}(\sim o_2),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\{(x,o_1,o_2), (\sim x,o_1,\sim o_2)\})$$
$$= \mathbf{m}_{R\leftarrow O_1}(\{(x,o_1,o_2), (\sim x,o_1,\sim o_2)\})\, \mathbf{m}_{R\leftarrow O_2}(\Theta_R)$$
$$= \mathbf{m}_{O_1}(o_1)\mathbf{m}_{O_2}(\Theta_{O_2}),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\sim x,\sim o_1,o_2)$$
$$= \mathbf{m}_{R\leftarrow O_1}(\{(\sim x,\sim o_1,o_2), (\sim x,\sim o_1,\sim o_2)\})\, \mathbf{m}_{R\leftarrow O_2}(\{(x,o_1,o_2), (\sim x,\sim o_1,o_2)\})$$
$$= \mathbf{m}_{O_1}(\sim o_1)\mathbf{m}_{O_2}(o_2),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\sim x,\sim o_1,\sim o_2)$$
$$= \mathbf{m}_{R\leftarrow O_1}(\{(\sim x,\sim o_1,o_2), (\sim x,\sim o_1,\sim o_2)\})\mathbf{m}_{R\leftarrow O_2}(\{(\sim x,o_1,\sim o_2),(\sim x,\sim o_1,\sim o_2)\})$$
$$= \mathbf{m}_{O_1}(\sim o_1)\mathbf{m}_{O_2}(\sim o_2),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$

$$= \mathbf{m}_{R\leftarrow O_1}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,\sim o_1,\sim o_2)\})\mathbf{m}_{R\leftarrow O_2}(\Theta_R)$$

$$= \mathbf{m}_{O_1}(\sim o_1)\mathbf{m}_{O_2}(\Theta_{O_2}),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\{(\chi,o_1,o_2), (\sim\chi,\sim o_1,o_2)\})$$

$$= \mathbf{m}_{R\leftarrow O_1}(\Theta_R)\mathbf{m}_{R\leftarrow O_2}(\{(\chi,o_1,o_2), (\sim\chi,\sim o_1,o_2)\})$$

$$= \mathbf{m}_{O_1}(\Theta_{O_1})\mathbf{m}_{O_2}(o_2),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\{(\sim\chi,o_1,\sim o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$

$$= \mathbf{m}_{R\leftarrow O_1}(\Theta_R)\mathbf{m}_{R\leftarrow O_2}(\{(\sim\chi,o_1,\sim o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$

$$= \mathbf{m}_{O_1}(\Theta_{O_1})\mathbf{m}_{O_2}(\sim o_2),$$

$$\mathbf{m}_{R\leftarrow\{O_1,O_2\}}(\Theta_R) = \mathbf{m}_{R\leftarrow O_1}(\Theta_R)\mathbf{m}_{R\leftarrow O_2}(\Theta_R) = \mathbf{m}_{O_1}(\Theta_{O_1})\mathbf{m}_{O_2}(\Theta_{O_2}).$$

After marginalizing the above **m**-values onto the frame of X, we obtain the **m**-values being sent to node X from node R which, in fact, is the result of **m**-values coming from nodes O_1 and O_2:[4]

$$\mathbf{m}_{X\leftarrow\{O_1,O_2\}}(\chi) = \mathbf{m}_{O_1}(o_1)\mathbf{m}_{O_2}(o_2), \quad (10)$$

$$\mathbf{m}_{X\leftarrow\{O_1,O_2\}}(\sim\chi) = \mathbf{m}_{O_1}(\sim o_1)\mathbf{m}_{O_2}(o_2) + \mathbf{m}_{O_1}(o_1)\mathbf{m}_{O_2}(\sim o_2) + \mathbf{m}_{O_1}(\sim o_1)\mathbf{m}_{O_2}(\sim o_2)$$
$$+ \mathbf{m}_{O_1}(\sim o_1)\mathbf{m}_{O_2}(\Theta_{O_2}) + \mathbf{m}_{O_1}(\Theta_{O_1})\mathbf{m}_{O_2}(\sim o_2), \quad (11)$$

$$\mathbf{m}_{X\leftarrow\{O_1,O_2\}}(\{\chi,\sim\chi\}) = \mathbf{m}_{O_1}(o_1)\mathbf{m}_{O_2}(\Theta_{O_2}) + \mathbf{m}_{O_1}(\Theta_{O_1})\mathbf{m}_{O_2}(o_2) + \mathbf{m}_{O_1}(\Theta_{O_1})\mathbf{m}_{O_2}(\Theta_{O_2}). \quad (12)$$

The above results are intuitive. Equation (10) suggests that if both sub-objectives have been met then the main objective is met as expected from the AND relationship that χ is true if and only if o_1 and o_2 are true. Equation (11) suggests that the main objective is not met under the following conditions: (i) when one of the sub-objectives is not met and the other sub-objective is met, (ii) when both sub-objectives are not met, and (iii) when one of the sub-objectives is not met and for the other we have no knowledge whether it is met. Equation (12) suggests that we have no knowledge about the main objective whether it is met under the following conditions: (i) when one of the sub-objectives has been met and for the other we have no knowledge that it is met, and (ii) when for both sub-objectives we have no knowledge that they are met.

Simplifying further, equations (10-12) can be rewritten as:

$$\mathbf{m}_{X\leftarrow\{O_1,O_2\}}(\chi) = \prod_{i=1}^{2} \mathbf{m}_{O_i}(o_i), \tag{13}$$

$$\mathbf{m}_{X\leftarrow \text{all O's}}(\sim\chi) = 1 - \prod_{i=1}^{2}[1 - \mathbf{m}_{O_i}(\sim o_i)], \tag{14}$$

$$\mathbf{m}_{X\leftarrow\{O_1,O_2\}}(\{\chi, \sim\chi\}) = 1 - \mathbf{m}_{X\leftarrow\{O_1,O_2\}}(\chi) - \mathbf{m}_{X\leftarrow\{O_1,O_2\}}(\sim\chi) \tag{15}$$

We have shown that equations (1–3) hold for the case of n = 2. By induction one can show that the results in (1–3) are true for any n. ♦

Proof of Proposition 2

Consider the propagation of **m**-values from nodes X and O_1 to node O_2 in Figure 4. The first step is to receive the **m**-values at the relational node R from nodes X and O_1. We combine these **m**-values with $\mathbf{m}_R$ using Dempster's rule and then marginalize it onto the frame of node O_2. This process yields the **m**-values at the sub-objective O_2. The **m**-values received by R from X is obtained by *vacuously* extending $\mathbf{m}_X$ to node R. The result is:

$$\mathbf{m}_{R\leftarrow X}(\chi, o_1, o_2) = \mathbf{m}_X(\chi),$$

$$\mathbf{m}_{R\leftarrow X}(\{(\sim\chi, \sim o_1, o_2), (\sim\chi, o_1, \sim o_2), (\sim\chi, \sim o_1, \sim o_2)\}) = \mathbf{m}_X(\sim\chi),$$

$$\mathbf{m}_{R\leftarrow X}(\Theta_R) = \mathbf{m}_X(\Theta_X). \tag{16}$$

m-values for all other subsets of Θ_R are zero. The **m**-values received by node R from O_1 are given in (8).

For propagating **m**-values to node O_2, we need to combine $\mathbf{m}_{R\leftarrow X}$, $\mathbf{m}_{R\leftarrow O_1}$, and $\mathbf{m}_R$, all defined at node R. Since $\mathbf{m}_R(\Theta_R) = 1$, the resultant **m**-values will be the same as the combination of $\mathbf{m}_{R\leftarrow X}$ and $\mathbf{m}_{R\leftarrow O_1}$. Combining $\mathbf{m}_{R\leftarrow X}$ and $\mathbf{m}_{R\leftarrow O_1}$ (Equations 8 and 16) yields $\mathbf{m}_{R\leftarrow\{X,O_1\}}$ as given below.

The renormalization constant K in Dempster's rule is given by:

$$K = 1 - \mathbf{m}_{R\leftarrow X}(\chi, o_1, o_2)\mathbf{m}_{R\leftarrow O_1}(\{(\sim\chi, \sim o_1, o_2), (\sim\chi, \sim o_1, \sim o_2)\})$$

$$= 1 - \mathbf{m}_X(\chi)\mathbf{m}_{O_1}(\sim o_1).$$

The non-zero values of $\mathbf{m}_{R\leftarrow\{X,O_1\}}$ are:

$$\mathbf{m}_{R\leftarrow\{X,O_1\}}(\chi,o_1,o_2)$$
$$= K^{-1}\, \mathbf{m}_{R\leftarrow X}(\chi,o_1,o_2)\, [\mathbf{m}_{R\leftarrow O_1}(\{(\chi,o_1,o_2), (\sim\chi,o_1,\sim o_2)\}) + \mathbf{m}_{R\leftarrow O_1}(\Theta_R)]$$
$$= K^{-1}\, \mathbf{m}_X(\chi)[\mathbf{m}_{O_1}(o_1) + \mathbf{m}_{O_1}(\Theta_{O_1})],$$

$$\mathbf{m}_{R\leftarrow\{X,O_1\}}(\sim\chi,o_1,\sim o_2)$$
$$= K^{-1}\, \mathbf{m}_{R\leftarrow X}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,o_1,\sim o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$
$$\times\, \mathbf{m}_{R\leftarrow O_1}(\{(\chi,o_1,o_2), (\sim\chi,o_1,\sim o_2)\})$$
$$= K^{-1}\, \mathbf{m}_X(\sim\chi)\mathbf{m}_{O_1}(o_1),$$

$$\mathbf{m}_{R\leftarrow\{X,O_1\}}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$
$$= K^{-1}\, [\mathbf{m}_{R\leftarrow X}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,o_1,\sim o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$
$$+ \mathbf{m}_{R\leftarrow X}(\Theta_R)]\, \mathbf{m}_{R\leftarrow O_1}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$
$$= K^{-1}\, [\mathbf{m}_X(\sim\chi) + \mathbf{m}_X(\Theta_X)]\, \mathbf{m}_{O_1}(\sim o_1),$$

$$\mathbf{m}_{R\leftarrow\{X,O_1\}}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,o_1,\sim o_2), (\sim\chi,\sim o_1,\sim o_2)\})$$
$$= K^{-1}\, [\mathbf{m}_{R\leftarrow X}(\{(\sim\chi,\sim o_1,o_2), (\sim\chi,o_1,\sim o_2), (\sim\chi,\sim o_1,\sim o_2)\})\, \mathbf{m}_{R\leftarrow O_1}(\Theta_R)]$$
$$= K^{-1}\, \mathbf{m}_X(\sim\chi)\, \mathbf{m}_{O_1}(\Theta_{O_1}),$$

$$\mathbf{m}_{R\leftarrow\{X,O_1\}}(\{(\chi,o_1,o_2), (\sim\chi,o_1,\sim o_2)\})$$
$$= K^{-1}\, \mathbf{m}_{R\leftarrow X}(\Theta_R)\, \mathbf{m}_{R\leftarrow O_1}(\{(\chi,o_1,o_2), (\sim\chi,o_1,\sim o_2)\})$$
$$= K^{-1}\, \mathbf{m}_X(\Theta_X)\mathbf{m}_{O_1}(o_1),$$

$$\mathbf{m}_{R\leftarrow\{X,O_1\}}(\Theta_R) = K^{-1}\, \mathbf{m}_{R\leftarrow X}(\Theta_R)\, \mathbf{m}_{R\leftarrow O_1}(\Theta_R) = K^{-1}\, \mathbf{m}_X(\Theta_X)\, \mathbf{m}_{O_1}(\Theta_{O_1}).$$

After marginalizing the above **m**-values, $\mathbf{m}_{R\leftarrow\{X,O_1\}}$, onto the frame of O_2 and simplifying, we obtain the following set of **m**-values propagated to node O_2 from nodes X and O_1:

$$\mathbf{m}_{O_2\leftarrow\{X,O_1\}}(o_2) = \mathbf{m}_{R\leftarrow\{X,O_1\}}(\chi,o_1,o_2) = K^{-1}\, \mathbf{m}_X(\chi)[\mathbf{m}_{O_1}(o_1) + \mathbf{m}_{O_1}(\Theta_{O_1})]$$
$$= K^{-1}\, \mathbf{m}_X(\chi)[1 - \mathbf{m}_{O_1}(\sim o_1)],$$

$$\mathbf{m}_{O2\leftarrow\{X,O1\}}(\sim o_2) = \mathbf{m}_{R\leftarrow\{X,O1\}}(\sim x, o_1, \sim o_2) = K^{-1}\, \mathbf{m}_X(\sim x)\mathbf{m}_{O1}(o_1),$$

$$\mathbf{m}_{O2\leftarrow\{X,O1\}}(\{o_2, \sim o_2\}) = 1 - \mathbf{m}_{O2\leftarrow\{X,O1\}}(o_2) - \mathbf{m}_{O2\leftarrow\{X,O1\}}(\sim o_2).$$

We have shown that equations (4–7) hold for the case of n = 2. By induction one can show that the results in (4–7) are true for any n. ♦

ENDNOTES

[1] In general, the frame of a node with three variables, X, O_1 and O_2, is the Cartesian product of the frames of the variables, i.e., $\Theta_R = \Theta_X \times \Theta_{O1} \times \Theta_{O2} = \{(x, o_1, o_2), (x, o_1, \sim o_2), (x, \sim o_1, o_2), (\sim x, o_1, o_2), (x, \sim o_1, \sim o_2)\ (\sim x, o_1, \sim o_2), (\sim x, \sim o_1, o_2), (\sim x, \sim o_1, \sim o_2)\}$. If we assume that O_1 and O_2 are related to X through an AND relationship, then we can represent this by a categorical bpa function **m** given by:

$$\mathbf{m}(\{(x, o_1, o_2), (\sim x, o_1, \sim o_2), (\sim x, \sim o_1, o_2), (\sim x, \sim o_1, \sim o_2)\}) = 1,$$

and all other **m**-values to be zero. Alternatively, we can represent the AND relationship by assuming that the frame of the relational node $\Theta_R = (\{(x, o_1, o_2), (\sim x, o_1, \sim o_2), (\sim x, \sim o_1, o_2), (\sim x, \sim o_1, \sim o_2)\})$. In the latter case, we don't need a bpa function to represent the AND relation. Thus the only bpa functions in an AND-tree are those that represent evidence.

[2] The American Accounting Association has established seven audit objectives: Validity, Completeness, Ownership, Valuation, Cutoff, Mechanical Accuracy, and Disclosure. The auditor collects evidence to establish that these objectives have been met for each account and thus establishes that each account is fairly stated (see, e.g., Arens and Loebbecke [7]).

[3] *Vacuous Extension:* Whenever a set of **m**-values is propagated from a smaller node (fewer variables) to a bigger node (more variables), the **m**-values are said to be *vacuously extended* onto the frame of the bigger node. As an illustration, suppose we have the following **m**-values on node O_1 with frame $\Theta_{O1} = \{o_1, \sim o_1\}$.

$$\mathbf{m}_{O1}(o_1) = 0.7,\ \mathbf{m}_{O1}(\sim o_1) = 0,\ \mathbf{m}_{O1}(\{o_1, \sim o_1\}) = 0.3$$

We want to vacuously extend them to a bigger node consisting of objectives O_1 and O_2. The entire frame of this combined node is obtained by multiplying the two individual frames, $\Theta_{O1} = \{o_1, \sim o_1\}$ and $\Theta_{O2} = \{o_2, \sim o_2\}$. The resulting frame is $\Theta_{O1O2} = \Theta_{O1} x \Theta_{O1} = \{(o_1, o_2),$

$(o_1,\sim o_2)$, $(\sim o_1,o_2)$, $(\sim o_1,\sim o_2)\}$. The vacuous extension of the above **m**-values from frame Θ_{O_1} = $\{o_1, \sim o_1\}$ to frame $\Theta_{O_1O_2}$ is as follows:

$$\mathbf{m}(\{(o_1,o_2), (o_1,\sim o_2)\}) = \mathbf{m}_{O_1}(o_1) = 0.7$$

$$\mathbf{m}(\Theta_{O_1O_2}) = \mathbf{m}_{O_1}(\Theta_{O_1}) = 0.3$$

and **m**-values for other subsets of $\Theta_{O_1O_2}$ are zero.

[4] *Marginalization:* Propagating **m**-values from a node to a smaller node is called *marginalization*. Let us consider the above example of Footnote 3 with slightly different **m**-values. Suppose we have the following **m**-values at $\Theta_{\{O_1,O_2\}}$ which is the frame of the combined nodes O_1 and O_2:

$$\mathbf{m}(o_1,o_2) = 0.4,$$

$$\mathbf{m}(\{(o_1,o_2), (o_1,\sim o_2)\}) = 0.2,$$

$$\mathbf{m}(\Theta_{O_1O_2}) = 0.4,$$

all other **m**-values are zero.

Let us first marginalize onto the frame $\Theta_{O_1} = \{o_1, \sim o_1\}$. Similar to marginalization of probabilities, we will sum all the **m**-values over the elements of frame $\Theta_{O_2} = \{o_2, \sim o_2\}$ for a given set of elements of frame $\Theta_{O_1} = \{o_1, \sim o_1\}$, i.e.,

$$\mathbf{m}(o_1) = \mathbf{m}(o_1,o_2) + \mathbf{m}(\{(o_1,o_2), (o_1,\sim o_2)\}) = 0.4 + 0.2 = 0.6,$$

$$\mathbf{m}(\sim o_1) = 0,$$

$$\mathbf{m}(\{o_1,\sim o_1\}) = \mathbf{m}(\Theta_{O_1O_2}) = 0.4.$$

Marginalizing onto the frame $\Theta_{O_2} = \{o_2, \sim o_2\}$ yields the following values:

$$\mathbf{m}(o_2) = \mathbf{m}(o_1,o_2) = 0.4,$$

$$\mathbf{m}(\sim o_2) = 0,$$

$$\mathbf{m}(\{o_2,\sim o_2\}) = \mathbf{m}(\{(o_1,o_2),(o_1,\sim o_2)\}) + \mathbf{m}(\Theta_{O_1O_2}) = 0.2 + 0.4 = 0.6.$$

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